THE INVARIANT OF n-PUNCTURED BALL TANGLES

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Abstract. Based on the Kauffman bracket at $A = e^{i\pi/4}$, we defined an invariant for a special type of n-punctured ball tangles. The invariant F^n takes values in the set $PM_{2\times 2^n}(\mathbb{Z})$ of 2×2^n matrices over $\mathbb{Z}$ modulo the scalar multiplication of ± 1. We provide the formula to compute the invariant of the $k_1 + \cdots + k_n$-punctured ball tangle composed of given $n, k_1, \ldots, k_n$-punctured ball tangles. Also, we define the horizontal and the vertical connect sums of punctured ball tangles and provide the formulas for their invariants from those of given punctured ball tangles. In addition, we introduce the elementary operations on the class $\boldsymbol{ST}$ of 1-punctured ball tangles, called spherical tangles. The elementary operations on $\boldsymbol{ST}$ induce the operations on $PM_{2\times 2}(\mathbb{Z})$, also called the elementary operations. We show that the group generated by the elementary operations on $PM_{2\times 2}(\mathbb{Z})$ is isomorphic to a Coxeter group.

1. Introduction

Throughout the paper, we work in either the smooth or the piecewise linear category. For basic terminologies of knot theory, see [1, 2].

We introduced a general definition of an n-punctured ball tangle and basic properties on them in [4]. However, our main interest still lies in a special type of n-punctured ball tangles, each boundary component of which intersects with the 1-dimensional proper submanifold at exactly 4 points. Hence, we restrict our scope to only such punctured ball tangles (Definition 2.1). In the case of $n = 0$, it corresponds exactly to Conway's notion of tangles in the 3-ball B^3 [3], and we call them *ball tangles*. Using the Kauffman bracket at $A = e^{i\pi/4}$, we defined an invariant for this special type of n-punctured ball tangles [4]. The invariant $F^n(T^n)$ for such an n-punctured ball tangle T^n is an element of the set $PM_{2\times 2^n}(\mathbb{Z})$ of 2×2^n matrices over $\mathbb{Z}$ modulo the scalar multiplication of ± 1. Specially, $F^0(T^0)$ is Krebes' invariant [5].

In this paper, we generalize the formula for the invariant of the ball tangle induced by an n-punctured ball tangle and n ball tangles. The invariant F^n behaves well under the operadic composition of n-punctured ball tangles. As a punctured ball tangle valued function, an n-punctured ball tangle T^n has the class of all punctured ball tangles as domain. When we put n many punctured ball tangles at the

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n holes of T^n, we have a new punctured ball tangle. That is, given an n-punctured ball tangle T^n and $k_1, \ldots, k_n$-punctured ball tangles $T^{k_1(1)}, \ldots, T^{k_n(n)}$, respectively, we consider the $k_1 + \cdots + k_n$-punctured ball tangle $T^n(T^{k_1(1)}, \ldots, T^{k_n(n)})$, where $n \in \mathbb{N}$ and $k_1, \ldots, k_n \in \mathbb{N} \cup \{0\}$. In this case, we show how to compute the invariant $F^{k_1+\cdots+k_n}(T^n(T^{k_1(1)}, \ldots, T^{k_n(n)}))$ if $F^n(T^n), F^{k_1}(T^{k_1(1)}), \ldots, F^{k_n}(T^{k_n(n)})$ are given (Theorem 3.2). Also, we consider the horizontal connect sum $T^{k_1(1)} +_h T^{k_2(2)}$ and the vertical connect sum $T^{k_1(1)} +_v T^{k_2(2)}$ of k_1 and k_2-punctured ball tangles $T^{k_1(1)}$ and $T^{k_2(2)}$, respectively, and provide the formulas for the invariants $F^{k_1+k_2}(T^{k_1(1)} +_h T^{k_2(2)})$ and $F^{k_1+k_2}(T^{k_1(1)} +_v T^{k_2(2)})$ from $F^{k_1}(T^{k_1(1)})$ and $F^{k_2}(T^{k_2(2)})$ (Theorem 3.3).

These generalizations can reduce much work when we try to compute the invariant for rather complicated punctured ball tangles. In order to compute the invariant for a given n-punctured ball tangle, we may successfully decompose it appropriately by already known ball tangles and punctured ball tangles in terms of compositions and connect sums. Then we will get the invariant of it by our formulas.

Finally, we introduce the elementary operations on the class $\boldsymbol{ST}$ of 1-punctured ball tangles, called spherical tangles. The elementary operations on $\boldsymbol{ST}$ induce the operations on $PM_{2\times 2}(\mathbb{Z})$, which is also called the elementary operations. We show that the group generated by the elementary operations on $PM_{2\times 2}(\mathbb{Z})$ is isomorphic to a Coxeter group (Theorem 4.6).

2. Preliminaries

In this section, we give a bunch of definitions and statements required for our main theorems. All of them come from our previous paper [4].

The notion of *tangles* was introduced by J. Conway [3] as the basic building blocks of links in the 3-dimensional sphere S^3. A tangle T is defined by a pair (B^3, T), where B^3 is a 3-dimensional closed ball and T is a 1-dimensional proper submanifold of B^3 with 2 non-circular components. The points in $\partial T \subset \partial B^3$ will be fixed all the time. Here, we considered holes inside the tangle such that if they are filled up with any tangles, we have a new tangle. In this sense, we define an *n-punctured ball tangle* slightly modified that in [4] to fit our purpose.

Definition 2.1. Let n be a nonnegative integer, and let H_0 be a closed 3-ball, and let $H_1, \ldots, H_n$ be pairwise disjoint closed 3-balls contained in the interior $\mathrm{Int}(H_0)$ of H_0. For each $k \in \{0, 1, \ldots, n\}$, take 4 distinct points $a_{k1}, a_{k2}, a_{k3}, a_{k4}$ of ∂H_k. Then a 1-dimensional proper submanifold T of $H_0 - \bigcup_{i=1}^{n} \mathrm{Int}(H_i)$ is called an n-punctured ball tangle with respect to $(H_k)_{0\le k\le n}$ and $((a_{k1}, a_{k2}, a_{k3}, a_{k4}))_{0\le k\le n}$ or, simply, an n-punctured ball tangle if $\partial T = \bigcup_{k=0}^{n} \{a_{k1}, a_{k2}, a_{k3}, a_{k4}\}$. Hence, $\partial T \cap \partial H_k = \{a_{k1}, a_{k2}, a_{k3}, a_{k4}\}$ for each $k \in \{0, 1, \ldots, n\}$.

Note that we can regard an n-punctured ball tangle T with respect to $(H_k)_{0\le k\le n}$ and $((a_{k1}, a_{k2}, a_{k3}, a_{k4}))_{0\le k\le n}$ as a 4-tuple $(n, (H_k)_{0\le k\le n}, ((a_{k1}, a_{k2}, a_{k3}, a_{k4}))_{0\le k\le n}, T)$.

Let $n \in \mathbb{N} \cup \{0\}$, and let $\boldsymbol{nPBT}$ be the class of all n-punctured ball tangles with respect to $(H_k)_{0\leq k\leq n}$ and $((a_{k1}, a_{k2}, a_{k3}, a_{k4}))_{0\leq k\leq n}$, and let $X = H_0 - \bigcup_{i=1}^{n} \mathrm{Int}(H_i)$. Define $\cong$ on $\boldsymbol{nPBT}$ by $T_1 \cong T_2$ if and only if there is a homeomorphism $h : X \to X$ such that $h|_{\partial X} = \mathrm{Id}_X|_{\partial X}$, $h(T_1) = T_2$, and h is isotopic to Id_X relative to the boundary ∂X for all $T_1, T_2 \in \boldsymbol{nPBT}$. Then $\cong$ is an equivalence relation on $\boldsymbol{nPBT}$, where Id_X is the identity map from X to X. n-punctured ball tangles T_1 and T_2 in $\boldsymbol{nPBT}$ are said to be equivalent or of the same isotopy type if $T_1 \cong T_2$. Also, for each n-punctured ball tangle T in $\boldsymbol{nPBT}$, the equivalence class of T with respect to $\cong$ is denoted by $[T]$. Without any confusion, we will also use T for $[T]$.

Like link diagrams, to deal with diagrams of n-punctured ball tangles in the same isotopy type, we need Reidemeister moves among them. For link diagrams or ball tangle diagrams, we have 3 kinds of Reidemeister moves. However, we need one and only one more kind of moves which are called the Reidemeister moves of type IV.

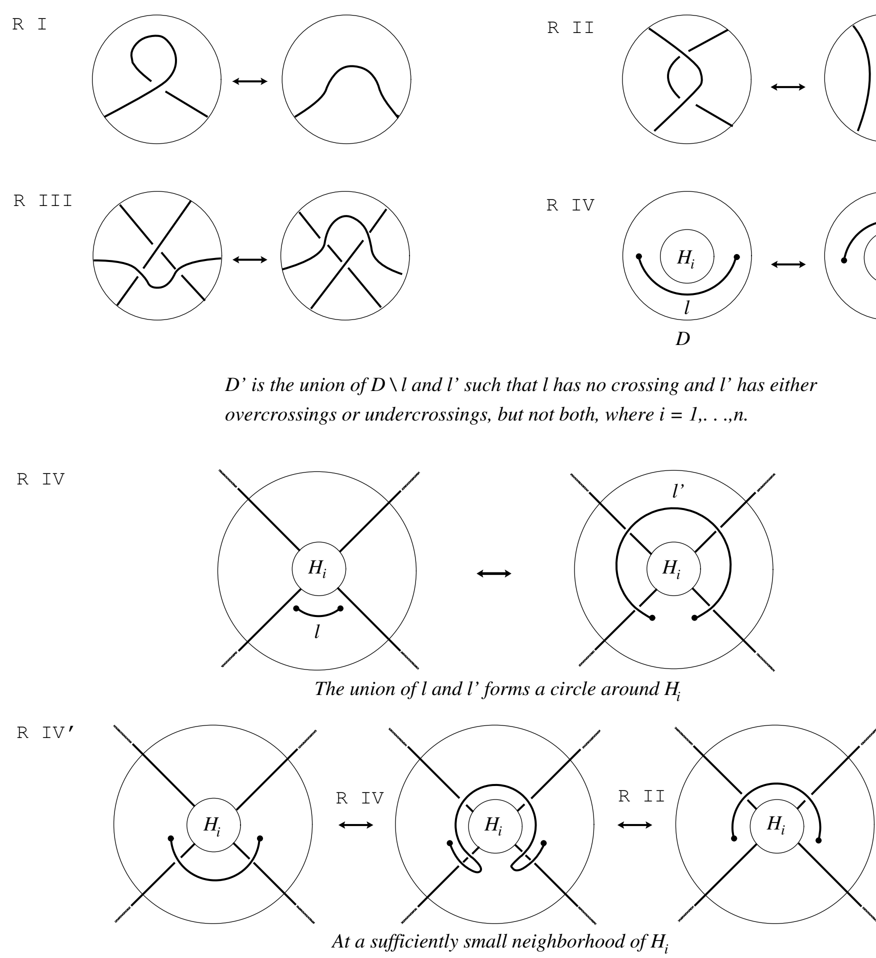


Figure 1. Tangle Reidemeister moves.

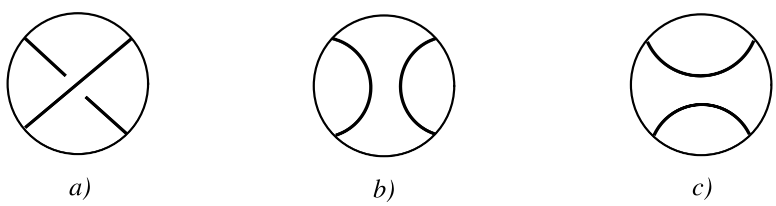
The Reidemeister moves for diagrams of n-punctured ball tangles are illustrated in Figure 1. Like link diagrams, tangle diagrams also have Reidemeister Theorem involving the Reidemeister moves of type IV. Let us call Reidemeister moves including type IV Tangle Reidemeister moves.

Theorem 2.2. *Let n be a nonnegative integer, and let D_1 and D_2 be diagrams of n-punctured ball tangles. Then $D_1 \cong D_2$ if and only if D_2 can be obtained from D_1 by a finite sequence of Tangle Reidemeister moves.*

There are many models for a class of n-punctured ball tangles. It is convenient to use normalized ones. One model for a class of n-punctured ball tangles is described in [4].

Our invariant is based on the Kauffman bracket at $A = e^{i\pi/4}$. Recall the Kauffman bracket is a regular isotopy invariant of link diagrams. That is, it will not be changed under Reidemeister moves of type II and III.

Note that a state σ of a link diagram L with n crossings $c_1, \ldots, c_n$ is regarded as a function $\sigma : \{c_1, \ldots, c_n\} \to \{A, B\}$, where A and B are the A-type and B-type splitting functions, respectively. Therefore, a link diagram L with n crossings has exactly 2^n states of it. Apply a state σ to L in order to change L to a diagram L_σ, called the resolution of L by σ, without any crossing.



a) a crossing c of L, b) the part of L_σ by $\sigma(c) = A$, c) the part of L_σ by $\sigma(c) = B$.
Figure 2. Two types of splitting of a crossing of L.

Definition 2.3. Let L be a link diagram. Then the Kauffman bracket $\langle L \rangle_A$, or simply, $\langle L \rangle$, is defined by

$$\langle L \rangle_A = \sum_{\sigma \in S} A^{\alpha(\sigma)} (A^{-1})^{\beta(\sigma)} (-A^2 - A^{-2})^{d(\sigma)-1},$$

where S is the set of all states of L, $\alpha(\sigma) = |\sigma^{-1}(A)|$, $\beta(\sigma) = |\sigma^{-1}(B)|$, and $d(\sigma)$ is the number of circles in L_σ.

We have the following skein relation of the Kauffman bracket.

Proposition 2.4. *Let L be a link diagram, and let c be a crossing of L. Then if L_A and L_B are link diagrams obtained from L by A-type splitting and B-type splitting only at c, respectively, then $\langle L \rangle = A\langle L_A \rangle + A^{-1}\langle L_B \rangle$.*

Proof. Suppose that S is the set of all states of L and $S_A = \{\sigma \in S | \sigma(c) = A\}$ and $S_B = \{\tau \in S | \tau(c) = B\}$. Then $\langle L \rangle = A \sum_{\sigma \in S_A} A^{\alpha(\sigma)-1}(A^{-1})^{\beta(\sigma)}(-A^2 - A^{-2})^{d(\sigma)-1} + A^{-1} \sum_{\tau \in S_B} A^{\alpha(\tau)}(A^{-1})^{\beta(\tau)-1}(-A^2 - A^{-2})^{d(\tau)-1} = A\langle L_A \rangle + A^{-1}\langle L_B \rangle$ because S is the disjoint union of S_A and S_B. This proves the proposition. □

Following [5], a state σ of a link diagram L is called a monocyclic state of L if $d(\sigma) = 1$. That is, we have only one circle when we remove all crossings of L by σ.

Also, it is proved in [5] that monocyclic states σ and σ' of L differ at an even number of crossings. The following lemma is a generalization of this statement.

Lemma 2.5 (J.-W. Chung and X.-S. Lin [4])**.** *Let L be a link diagram. Then states σ and σ' of L are of the same parity, i.e., $d(\sigma) \equiv d(\sigma') \bmod 2$, if and only if σ and σ' differ at an even number of crossings, where $d(\sigma)$ and $d(\sigma')$ are the numbers of circles in L_σ and $L_{\sigma'}$, respectively.*

Proof. Let σ be a state of a link diagram L with n crossings $c_1, \ldots, c_n$. Change the value of σ at only one crossing c_i to get another state σ_i and observe what happens to $d(\sigma_i)$, where $1 \le i \le n$. We claim that σ and σ_i have different parities, more precisely, $d(\sigma) = d(\sigma_i) \pm 1$. Hence, we will have $d(\sigma) \equiv d(\sigma_i) + 1 \bmod 2$. Now, to consider $\sigma(c_i)$ and $\sigma_i(c_i)$, take a sufficiently small neighborhood B_i at the projection of c_i so that the intersection of $\mathrm{Int}(B_i)$ and the set of all double points of L is the projection of c_i and the intersection of ∂B_i and the projection of L has exactly 4 points on the projection plane of L which are not double points of L.

Case 1. If these 4 points are on a circle in L_σ, then

$$d(\sigma_i) = d(\sigma) + 1.$$

Case 2. If two of 4 points are on a circle and the other points are on another circle in L_σ, then

$$d(\sigma_i) = d(\sigma) - 1.$$

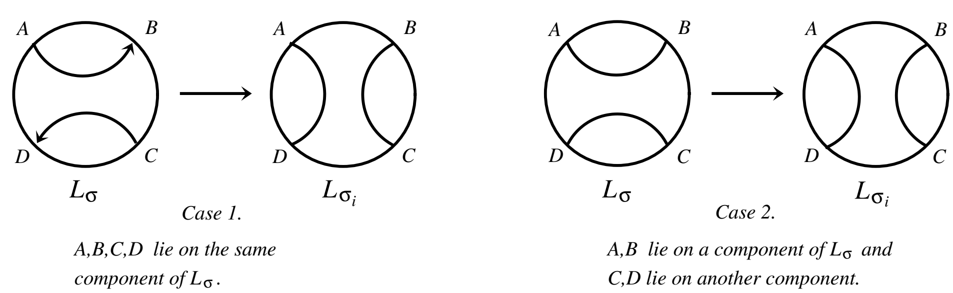


Figure 3. Proof of Lemma 2.5.

Now, it is easy to show the lemma. Suppose that σ and σ' are states of L which differ at k crossings of L for some $k \in \{0, 1, \dots, n\}$. Then $d(\sigma') \equiv d(\sigma) + k \bmod 2$. If $d(\sigma) \equiv d(\sigma') \bmod 2$, then k is even. Conversely, if $d(\sigma) \equiv d(\sigma') + 1 \bmod 2$, then $k+1$ is even, that is, k is odd. This proves the lemma. $\square$

Suppose that $A = e^{i\pi/4}$. Then $-A^2 - A^{-2} = 0$. Therefore,

$$\langle L \rangle = \sum_{\sigma \in M} A^{\alpha(\sigma)-\beta(\sigma)},$$

where M is the set of all monocyclic states of L.

From now on, we use only the Kauffman brackets at $A = e^{i\pi/4}$. Note that, since $|A| = 1$, the determinant $|\langle L \rangle|$ of L is an isotopy invariant.

Lemma 2.6. *If L is a link diagram, then there are $p \in \mathbb{Z}$ and $u \in \mathbb{C}$ such that $u^8 = 1$ and $\langle L \rangle = pu$.*

The following notations throughout the rest of the paper:

- $\Phi = \{z \in \mathbb{C} \,|\, z^8 = 1\} = \{A^k \,|\, k \in \mathbb{Z}\}$ and $\mathbb{Z}\Phi = \{kz \,|\, k \in \mathbb{Z}, z \in \Phi\}$.
- $PM_{m\times n}(\mathbb{Z})$ is the quotient of $M_{m\times n}(\mathbb{Z})$ under the scalar multiplication by ± 1.
- $\boldsymbol{BT}$ is the class of diagrams of 0-punctured ball tangles (i.e. ball tangles).
- $\boldsymbol{ST}$ is the class of diagrams of 1-punctured ball tangles (they will be called spherical tangles).

Proposition 2.7. *If $a, b, k, l \in \mathbb{Z}$, then $aA^k + bA^l \in \mathbb{Z}\Phi$ if and only if $ab = 0$ or $k \equiv l$ mod 4.*

Given a ball tangle diagram B, consider 2 kinds of closures as in Figure 4. The link diagrams B_1 and B_2 are called the numerator closure and the denominator closure of B, respectively. A monocyclic state of B_1 is called a numerator state of B and that of B_2 is a denominator state of B.

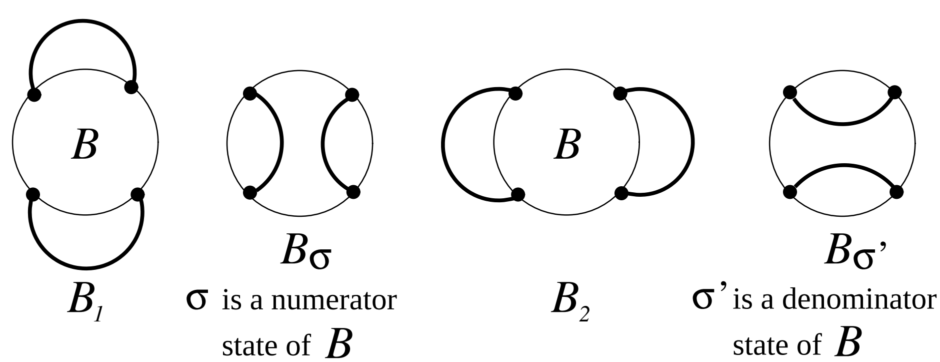


Figure 4. The numerator closure B_1 and the denominator closure B_2.

Notice that a numerator state σ and a denominator state σ' of a ball tangle diagram B differ at an odd number of crossings. To see this, we think of a link diagram L such that B embeds in L and L has one and only one more crossing c at the outside of the ball containing B and L has no self-twist at the outside of the ball. We have two monocyclic states of L from the numerator state σ and the denominator state σ', respectively, which differ at c. Hence, σ and σ' differ at an odd number of crossings. Without loss of generality, we may assume that

$$\langle L \rangle = A\langle B_1 \rangle + A^{-1}\langle B_2 \rangle \in \mathbb{Z}\Phi.$$

If $\langle B_1 \rangle = pA^k$ and $\langle B_2 \rangle = qA^l$, by Proposition 2.7, we have $l \equiv k + 2 \bmod 4$. Hence, there is a unique $(\alpha, \beta) \in \mathbb{Z}^2$ such that

$$\left\{ \begin{pmatrix} z\langle B_1 \rangle \\ iz\langle B_2 \rangle \end{pmatrix} \mid z \in \Phi \right\} \cap M_{2\times 1}(\mathbb{Z}) = \left\{ \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} -\alpha \\ -\beta \end{pmatrix} \right\} := \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in PM_{2\times 1}(\mathbb{Z}).$$

Definition 2.8. (Krebes [5]) Define $f : \boldsymbol{BT} \to PM_{2\times 1}(\mathbb{Z})$ by

$$f(B) = \left\{ \begin{pmatrix} z\langle B_1 \rangle \\ iz\langle B_2 \rangle \end{pmatrix} \mid z \in \Phi \right\} \cap M_{2\times 1}(\mathbb{Z}) \in PM_{2\times 1}(\mathbb{Z})$$

for each $B \in \boldsymbol{BT}$. This is Krebes' tangle invariant.

Let n be a positive integer. Then an n-punctured ball tangle T^n with $(H_k)_{0\leq k\leq n}$ can be regarded as an n-variable function $T^n : \boldsymbol{A}_1 \times \cdots \times \boldsymbol{A}_n \to \boldsymbol{T}$ defined as $T^n(X_1, \ldots, X_n)$ is a tangle filled up in the i-th hole H_i of T^n by $X_i \in \boldsymbol{A}_i$ for each $i \in \{1, \ldots, n\}$, where $\boldsymbol{A}_i$ is a class of t_i-punctured ball tangles for each $i \in \{1, \ldots, n\}$ and $\boldsymbol{T}$ is a class of tangles. However, this representation of n-punctured ball tangles as n-variable functions is not perfect in the sense that n-punctured ball tangles are equivalent only if they induce the same function. On the other hand, n-punctured ball tangles which induce the same function need not be equivalent. That is, we can say that tangles are stronger than functions.

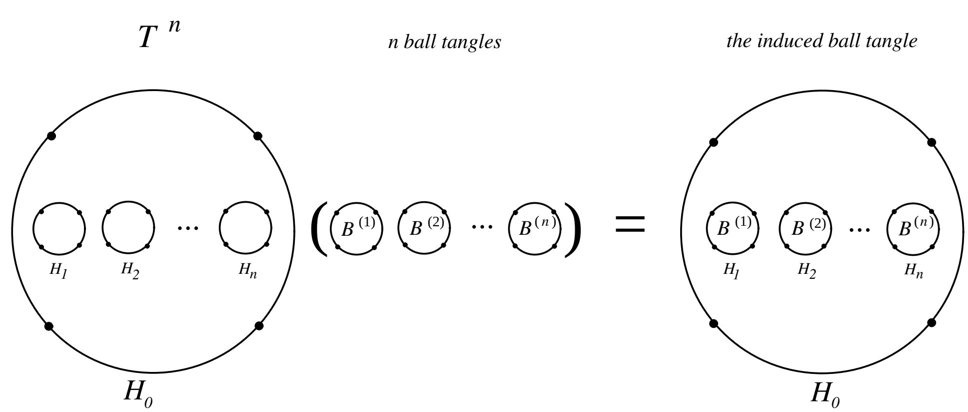


Figure 5. The induced ball tangle $T^n(B^{(1)}, \cdots, B^{(n)})$ by T^n and $B^{(1)}, \cdots, B^{(n)}$.

Roughly speaking, the class of n-punctured ball tangles as only n-variable functions gives us an *operad*, a mathematical device which describes algebraic structure of many varieties and in various categories. See [6].

First of all, to construct the invariant F^n of n-punctured ball tangle T^n, let us regard T^n as a 'hole-filling function', in sense described as above $T^n : \boldsymbol{BT}^n \to \boldsymbol{BT}$, where $\boldsymbol{BT}^n = \boldsymbol{BT}_1 \times \cdots \times \boldsymbol{BT}_n$ with $\boldsymbol{BT}_1 = \cdots = \boldsymbol{BT}_n = \boldsymbol{BT}$ (Figure 5).

Also, to construct our invariant of n-punctured ball tangles, we need to use some quite complicated notations. Let us start with a gentle introduction to our notations:

(1) For a diagram of 0-punctured ball tangle T^0 (a ball tangle), we can produce 2 links T^0_1 and T^0_2, which are the numerator closure and the denominator closure of T^0, respectively.

(2) For a diagram of 1-punctured ball tangle T^1 (a spherical tangle), we can produce 2^{1+1} links $T^1_{1(1)}, T^1_{1(2)}; T^1_{2(1)}, T^1_{2(2)}$, where the subscript $1(1)$ means to take the numerator closure of T with its hole filled by the fundamental tangle 1.

(3) For a diagram of 2-punctured ball tangle T^2, we can produce 2^{2+1} links $T^2_{1(11)}$, $T^2_{1(12)}, T^2_{1(21)}, T^2_{1(22)}; T^2_{2(11)}, T^2_{2(12)}, T^2_{2(21)}, T^2_{2(22)}$.

If n is a positive integer, $J_1 = \cdots = J_n = \{1, 2\}$, and $J(n) = \prod_{k=1}^n J_k$, then $J(n)$ is linearly ordered by a dictionary order, or lexicographic order, consisting of 2^n ordered n-tuples each of whose components is either 1 or 2. That is, if $x, y \in J(n)$ and $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, then $x < y$ if and only if $x_1 < y_1$ or there is $k \in \{1, \ldots, n-1\}$ such that $x_1 = y_1, \ldots, x_k = y_k, x_{k+1} < y_{k+1}$.

(4) $J(n) = \{\alpha^n_i | 1 \le i \le 2^n\}$ and $\alpha^n_1 < \alpha^n_2 < \cdots < \alpha^n_{2^n}$, where $<$ is the dictionary order on $J(n)$. Hence, α^n_1 is the least element $(1, 1, \ldots, 1)$ and $\alpha^n_{2^n}$ is the greatest element $(2, 2, \ldots, 2)$ of $J(n)$. Let us denote $\alpha^n_i = (\alpha^n_{i1}, \ldots, \alpha^n_{in})$ for each $i \in \{1, \ldots, 2^n\}$.

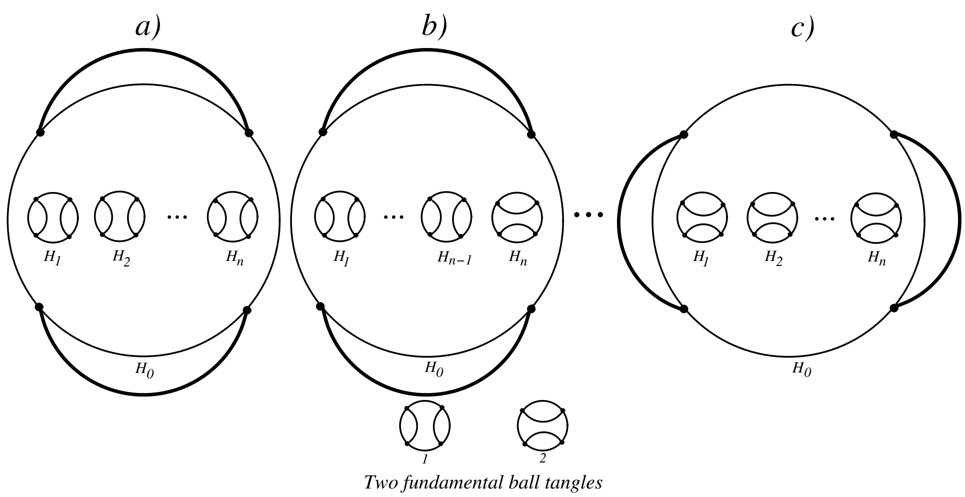


Figure 6. The closures of T^n, a) $T^n_{1\alpha^n_1}$, b) $T^n_{1\alpha^n_2}$, c) $T^n_{2\alpha^n_{2^n}}$.

(5) For a diagram of n-punctured ball tangle T^n, we can produce 2^{n+1} links $T^n_{1\alpha^n_1}, \ldots, T^n_{1\alpha^n_{2^n}}; T^n_{2\alpha^n_1}, \ldots, T^n_{2\alpha^n_{2^n}}$.

(6) The sequence $(a_n)_{n\geq 0} = ((t_k)_{1\leq k\leq 2^n})_{n\geq 0}$ is defined recursively as follows:

1) $a_0 = (0)$;

2) If $a_{k-1} = (t_1, \ldots, t_{2^{k-1}})$, then $a_k = (t_1, \ldots, t_{2^{k-1}}, t_1 + 1, \ldots, t_{2^{k-1}} + 1)$ for each $k \in N$. Note that $t_{2^n} = n$ for each $n \in \mathbb{N} \cup \{0\}$.

Now, we define our invariant of n-punctured ball tangles inductively.

Theorem 2.9. *For each $n \in \mathbb{N}$, define $F^n : \boldsymbol{nPBT} \to PM_{2\times 2^n}(\mathbb{Z})$ by*

$$F^n(T^n) = \left\{ \begin{pmatrix} (-i)^{t_1} z\langle T^n_{1\alpha^n_1}\rangle & \cdots & (-i)^{t_{2^n}} z\langle T^n_{1\alpha^n_{2^n}}\rangle \\ (-i)^{t_1} iz\langle T^n_{2\alpha^n_1}\rangle & \cdots & (-i)^{t_{2^n}} iz\langle T^n_{2\alpha^n_{2^n}}\rangle \end{pmatrix} \mid z \in \Phi \right\} \cap M_{2\times 2^n}(\mathbb{Z})$$

for each $T^n \in \boldsymbol{nPBT}$. Then F^n is an isotopy invariant of n-punctured ball tangle diagrams. In particular, F^0 is Krebes' ball tangle invariant f.

Definition 2.10. For each nonnegative integer n, F^n is called the n-punctured ball tangle invariant, simply, the n-punctured tangle invariant.

In order to think of n-punctured ball tangle T^n as a 'hole-filling function', we define a function which makes a dictionary order on complex numbers.

Let n be a positive integer, and let $(k_1, \ldots, k_n)$ be an n-tuple of positive integers, and let $J(n, k_1, \ldots, k_n) = \prod_{i=1}^n I_{k_i}$. Then $J(n, k_1, \ldots, k_n)$ is linearly ordered by a dictionary order, where $I_k = \{1, \ldots, k\}$ for each $k \in \mathbb{N}$.

(4*) $J(n, k_1, \ldots, k_n) = \{\alpha_i^{n,k_1,\ldots,k_n} | 1 \leq i \leq k_1 \cdots k_n\}$ and $\alpha_1^{n,k_1,\ldots,k_n} < \cdots < \alpha_{k_1\cdots k_n}^{n,k_1,\ldots,k_n}$, where $<$ is the dictionary order on $J(n, k_1, \ldots, k_n)$ and $\alpha_1^{n,k_1,\ldots,k_n}$ is the least element $(1, 1, \ldots, 1)$ and $\alpha_{k_1\cdots k_n}^{n,k_1,\ldots,k_n}$ is the greatest element $(k_1, k_2, \ldots, k_n)$ of $J(n, k_1, \ldots, k_n)$. Let us denote $\alpha_i^{n,k_1,\ldots,k_n} = (\alpha_{i1}^{n,k_1,\ldots,k_n}, \ldots, \alpha_{in}^{n,k_1,\ldots,k_n})$ for each $i \in \{1, \ldots, k_1 \cdots k_n\}$.

Definition 2.11. For each $n \in \mathbb{N}$ and n-tuple $(k_1, \ldots, k_n)$ of positive integers, define

$$\xi^{n,k_1,\ldots,k_n} : \mathbb{C}^{k_1} \times \cdots \times \mathbb{C}^{k_n} \to \mathbb{C}^{k_1\cdots k_n}$$

by

$$\xi^{n,k_1,\ldots,k_n}((v_1^1, \ldots, v_{k_1}^1), \ldots, (v_1^n, \ldots, v_{k_n}^n)) = \left(\prod_{j=1}^n v^j_{\alpha_{1j}^{n,k_1,\ldots,k_n}}, \ldots, \prod_{j=1}^n v^j_{\alpha_{k_1\cdots k_n j}^{n,k_1,\ldots,k_n}} \right)$$

for all $(v_1^1, \ldots, v_{k_1}^1) \in \mathbb{C}^{k_1}, \ldots, (v_1^n, \ldots, v_{k_n}^n) \in \mathbb{C}^{k_n}$. Then $\xi^{n,k_1,\ldots,k_n}$ is well-defined and called the dictionary order function on $\mathbb{C}$ with respect to $k_1, \ldots, k_n$. Also, the i-th projection of $\xi^{n,k_1,\ldots,k_n}$ is denoted by $\xi_i^{n,k_1,\ldots,k_n}$ for each $i \in \{1, \ldots, k_1 \cdots k_n\}$. In particular, we simply denote $\xi^{n,k_1,\ldots,k_n}$ by ξ^n when $k_1 = \cdots = k_n = 2$.

Denote by $\mathbb{C}^{k\dagger}$ the k-dimensional column vector space over $\mathbb{C}$, so the map

$$(v_1, \ldots, v_k) \mapsto (v_1, \ldots, v_k)^\dagger : \mathbb{C}^k \longrightarrow \mathbb{C}^{k\dagger}$$

is to transpose row vectors to column vectors. Let $P\mathbb{C}^{k\dagger} = \mathbb{C}^{k\dagger}/\pm 1$. If $(v_1, \ldots, v_k)^\dagger \in \mathbb{C}^{k\dagger}$, then we denote by

$$[v_1, \ldots, v_k]^\dagger = \{(v_1, \ldots, v_k)^\dagger, (-v_1, \ldots, -v_k)^\dagger\}$$

the corresponding element in $P\mathbb{C}^{k\dagger}$.

Remark that we may extend the above notation to matrices modulo ± 1. Under this extension, matrix multiplication is well-defined. That is, if A and B are matrices and AB is defined, then $[A][B] = [A][-B] = [-A][B] = [-A][-B] = [-AB] = [AB]$.

Lemma 2.12. *For each $n \in \mathbb{N}$ and n-tuple $(k_1, \ldots, k_n)$ of positive integers, define*

$$[\xi^{n,k_1,\ldots,k_n}] : P\mathbb{C}^{k_1\dagger} \times \cdots \times P\mathbb{C}^{k_n\dagger} \longrightarrow P\mathbb{C}^{k_1\cdots k_n\dagger}$$

by

$$[\xi^{n,k_1,\ldots,k_n}]\left(\begin{bmatrix} v_1^1 \\ \cdot \\ \cdot \\ \cdot \\ v_{k_1}^1 \end{bmatrix}, \ldots, \begin{bmatrix} v_1^n \\ \cdot \\ \cdot \\ \cdot \\ v_{k_n}^n \end{bmatrix}\right) = \begin{bmatrix} \prod_{j=1}^n v^j_{\alpha_{1j}^{n,k_1,\ldots,k_n}} \\ \cdot \\ \cdot \\ \cdot \\ \prod_{j=1}^n v^j_{\alpha_{k_1\cdots k_n j}^{n,k_1,\ldots,k_n}} \end{bmatrix}$$

for all $(v_1^1, \ldots, v_{k_1}^1) \in \mathbb{C}^{k_1}, \ldots, (v_1^n, \ldots, v_{k_n}^n) \in \mathbb{C}^{k_n}$. Then $[\xi^{n,k_1,\ldots,k_n}]$ is well-defined and called the dictionary order function induced by $\xi^{n,k_1,\ldots,k_n}$.

As another notation, if L is a link diagram and T^n is a diagram of n-punctured ball tangle for some $n \in \mathbb{N} \cup \{0\}$, then the sets of all crossings of L and T^n are denoted by $c(L)$ and $c(T^n)$, respectively.

Lemma 2.13. *If $n \in \mathbb{N}$ and T^n is an n-punctured ball tangle diagram and $B^{(1)}, \ldots, B^{(n)}$ are ball tangle diagrams, then*

$$\begin{pmatrix} \langle T^n(B^{(1)}, \ldots, B^{(n)})_1 \rangle \\ \langle T^n(B^{(1)}, \ldots, B^{(n)})_2 \rangle \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{2^n} \langle T^n_{1\alpha_i^n} \rangle \langle B^{(1)}_{\alpha_{i1}^n} \rangle \cdots \langle B^{(n)}_{\alpha_{in}^n} \rangle \\ \sum_{i=1}^{2^n} \langle T^n_{2\alpha_i^n} \rangle \langle B^{(1)}_{\alpha_{i1}^n} \rangle \cdots \langle B^{(n)}_{\alpha_{in}^n} \rangle \end{pmatrix}.$$

Theorem 2.14 (J.-W. Chung and X.-S. Lin [4])**.** *For each $n \in \mathbb{N}$, F^n is an n-punctured ball tangle invariant such that*

$$F^0(T^n(B^{(1)}, \ldots, B^{(n)})) = F^n(T^n)[\xi^n](F^0(B^{(1)}), \ldots, F^0(B^{(n)}))$$

for all $B^{(1)}, \ldots, B^{(n)} \in \boldsymbol{BT}$.

Proof. Suppose that T^n is an n-punctured ball tangle such that $F^n(T^n) = [zX(T^n)]$ for some $z \in \Phi$ and $B^{(1)}, \ldots, B^{(n)}$ are ball tangles such that

$$F^0(B^{(1)}) = \begin{bmatrix} z_1\langle B_1^{(1)}\rangle \\ iz_1\langle B_2^{(1)}\rangle \end{bmatrix}, \ldots, F^0(B^{(n)}) = \begin{bmatrix} z_n\langle B_1^{(n)}\rangle \\ iz_n\langle B_2^{(n)}\rangle \end{bmatrix}$$

for some $z_1, \ldots, z_n \in \Phi$, where $\langle B_1^{(i)}\rangle$ and $\langle B_2^{(i)}\rangle$ are the numerator closure and the denominator closure of $B^{(i)}$, respectively, for each $i \in \{1, \ldots, n\}$. Then

$$\begin{aligned}
&F^n(T^n)[\xi^n](F^0(B^{(1)}), \ldots, F^0(B^{(n)})) \\
&= \left[\begin{pmatrix} (-i)^{t_1} z\langle T^n_{1\alpha_1^n}\rangle & \cdots & (-i)^{t_{2^n}} z\langle T^n_{1\alpha_{2^n}^n}\rangle \\ (-i)^{t_1} iz\langle T^n_{2\alpha_1^n}\rangle & \cdots & (-i)^{t_{2^n}} iz\langle T^n_{2\alpha_{2^n}^n}\rangle \end{pmatrix} \begin{pmatrix} i^{t_1} z_1 \cdots z_n \langle B^{(1)}_{\alpha^n_{11}}\rangle \cdots \langle B^{(n)}_{\alpha^n_{1n}}\rangle \\ \cdot \\ \cdot \\ \cdot \\ i^{t_{2^n}} z_1 \cdots z_n \langle B^{(1)}_{\alpha^n_{2^n 1}}\rangle \cdots \langle B^{(n)}_{\alpha^n_{2^n n}}\rangle \end{pmatrix} \right] \\
&= \begin{bmatrix} zz_1\cdots z_n(\langle T^n_{1\alpha_1^n}\rangle\langle B^{(1)}_{\alpha^n_{11}}\rangle \cdots \langle B^{(n)}_{\alpha^n_{1n}}\rangle + \cdots\cdots + \langle T^n_{1\alpha_{2^n}^n}\rangle\langle B^{(1)}_{\alpha^n_{2^n 1}}\rangle \cdots \langle B^{(n)}_{\alpha^n_{2^n n}}\rangle) \\ izz_1\cdots z_n(\langle T^n_{2\alpha_1^n}\rangle\langle B^{(1)}_{\alpha^n_{11}}\rangle \cdots \langle B^{(n)}_{\alpha^n_{1n}}\rangle + \cdots\cdots + \langle T^n_{2\alpha_{2^n}^n}\rangle\langle B^{(1)}_{\alpha^n_{2^n 1}}\rangle \cdots \langle B^{(n)}_{\alpha^n_{2^n n}}\rangle) \end{bmatrix} \\
&= \begin{bmatrix} zz_1\cdots z_n \sum_{i=1}^{2^n}\langle T^n_{1\alpha_i^n}\rangle\langle B^{(1)}_{\alpha^n_{i1}}\rangle \cdots \langle B^{(n)}_{\alpha^n_{in}}\rangle \\ izz_1\cdots z_n \sum_{i=1}^{2^n}\langle T^n_{2\alpha_i^n}\rangle\langle B^{(1)}_{\alpha^n_{i1}}\rangle \cdots \langle B^{(n)}_{\alpha^n_{in}}\rangle \end{bmatrix} = \begin{bmatrix} zz_1\cdots z_n\langle T^n(B^{(1)}, \ldots, B^{(n)})_1\rangle \\ izz_1\cdots z_n\langle T^n(B^{(1)}, \ldots, B^{(n)})_2\rangle \end{bmatrix} \\
&= F^0(T^n(B^{(1)}, \ldots, B^{(n)}))
\end{aligned}$$

by Lemma 2.13. □

3. Generalized formulas for invariant of n-punctured ball tangles

Notice that an n-punctured ball tangle T^n may be regarded as an n variable function about not only 0-punctured ball tangles but also various punctured ball tangles. Given an n-punctured ball tangle diagram T^n and $k_1, \ldots, k_n$-punctured ball tangle diagrams $T^{k_1(1)}, \ldots, T^{k_n(n)}$, respectively, we consider the induced $k_1 + \cdots + k_n$-punctured ball tangle diagram $T^n(T^{k_1(1)}, \ldots, T^{k_n(n)})$, where $n \in \mathbb{N}$ and $k_1, \ldots, k_n \in \mathbb{N} \cup \{0\}$. We show how to calculate the invariant $F^{k_1+\cdots+k_n}(T^n(T^{k_1(1)}, \ldots, T^{k_n(n)}))$ of it if $F^n(T^n), F^{k_1}(T^{k_1(1)}), \ldots, F^{k_n}(T^{k_n(n)})$ are given (Theorem 3.2). On the other hand, we consider the horizontal connect sum $T^{k_1(1)} +_h T^{k_2(2)}$ and the vertical connect sum $T^{k_1(1)} +_v T^{k_2(2)}$ of k_1 and k_2-punctured ball tangles $T^{k_1(1)}$ and $T^{k_2(2)}$, respectively, and provide the formulas for the invariants $F^{k_1+k_2}(T^{k_1(1)} +_h T^{k_2(2)})$ and $F^{k_1+k_2}(T^{k_1(1)} +_v T^{k_2(2)})$ from $F^{k_1}(T^{k_1(1)})$ and $F^{k_2}(T^{k_2(2)})$ (Theorem 3.3). To prove these two generalized formulas, we require a statement from 'Projective Linear Algebra' (Lemma 3.1). Let us start from the following notations:

Let $n \in \mathbb{N}$. Then

(1) $e_i^n = \begin{bmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_{2^n} \end{bmatrix}$ such that $v_i = 1$ and $v_j = 0$ if $j \neq i$ for each $i \in \{1, \dots, 2^n\}$.

In particular, $e_1^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Hence, $e_i^n = [\xi^n](e^1_{\alpha^n_{i1}}, \dots, e^1_{\alpha^n_{in}})$ for each $i \in \{1, \dots, 2^n\}$.

(2) $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(3) E_j^n is the set of all $[\xi^n](y_1, \dots, y_n)$ such that j components of $(y_1, \dots, y_n)$ are x and each of the others is e_1^1 or e_2^1 for each $j \in \{0, 1, \dots, n\}$. In particular,

$$E_0^n = \{[\xi^n](e^1_{\alpha^n_{i1}}, \dots, e^1_{\alpha^n_{in}}) | i \in \{1, \dots, 2^n\}\}$$

and

$$E_n^n = \{[\xi^n](y_1, \dots, y_n) | y_1 = \cdots = y_n = x\}.$$

Notice that $\{E_0^n, E_1^n, \dots, E_n^n\}$ is pairwise disjoint and $|E_j^n| = {}_nC_j\, 2^{n-j}$ for each $j \in \{0, 1, \dots, n\}$, where ${}_nC_j = \frac{n!}{(n-j)!j\,!}$. Hence,

$$|\coprod_{j=0}^{n} E_j^n| = {}_nC_0\, 2^n + {}_nC_1\, 2^{n-1} + \cdots + {}_nC_{n-1}\, 2^1 + {}_nC_n\, 2^0 = (2+1)^n = 3^n.$$

Note that

$$\coprod_{j=0}^{n} E_j^n = [\xi^n](\{e_1^1, e_2^1, x\}^n).$$

For example, when $n = 3$, we have

$$E_0^3 = \{e_1^3, e_2^3, e_3^3, e_4^3, e_5^3, e_6^3, e_7^3, e_8^3\},$$

$$\begin{aligned} E_1^3 = \{ & [1\,0\,0\,0\,1\,0\,0\,0]^\dagger, [0\,1\,0\,0\,0\,1\,0\,0]^\dagger, [0\,0\,1\,0\,0\,0\,1\,0]^\dagger, [0\,0\,0\,1\,0\,0\,0\,1]^\dagger, \\ & [1\,0\,1\,0\,0\,0\,0\,0]^\dagger, [0\,1\,0\,1\,0\,0\,0\,0]^\dagger, [0\,0\,0\,0\,1\,0\,1\,0]^\dagger, [0\,0\,0\,0\,0\,1\,0\,1]^\dagger, \\ & [1\,1\,0\,0\,0\,0\,0\,0]^\dagger, [0\,0\,1\,1\,0\,0\,0\,0]^\dagger, [0\,0\,0\,0\,1\,1\,0\,0]^\dagger, [0\,0\,0\,0\,0\,0\,1\,1]^\dagger\}, \end{aligned}$$

$$\begin{aligned} E_2^3 = \{ & [1\,0\,1\,0\,1\,0\,1\,0]^\dagger, [0\,1\,0\,1\,0\,1\,0\,1]^\dagger, \\ & [1\,1\,0\,0\,1\,1\,0\,0]^\dagger, [0\,0\,1\,1\,0\,0\,1\,1]^\dagger, \\ & [1\,1\,1\,1\,0\,0\,0\,0]^\dagger, [0\,0\,0\,0\,1\,1\,1\,1]^\dagger\}, \end{aligned}$$

$$E_3^3 = \{[1\,1\,1\,1\,1\,1\,1\,1]^\dagger\}.$$

Now, we have the following lemma which supports our main theorems.

Lemma 3.1. *If $n \in \mathbb{N}$ and $A, B \in PM_{2\times 2^n}(\mathbb{Z})$ and $AX = BX$ for each $X \in [\xi^n](\{e_1^1, e_2^1, x\}^n)$, then $A = B$.*

Proof. We prove the statement by induction on $n \in \mathbb{N}$.

Step 1. We show that the statement is true for $n = 1$.

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, and let $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$. Since $E_0^1 = \{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$ and $E_1^1 = \{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$ and $AX = BX$ for each $X \in [\xi^1](\{e_1^1, e_2^1, x\}^1)$, $\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}$ and $\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix}$ and $\begin{bmatrix} a_{11} + a_{12} \\ a_{21} + a_{22} \end{bmatrix} = \begin{bmatrix} b_{11} + b_{12} \\ b_{21} + b_{22} \end{bmatrix}$. Hence, $\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \epsilon \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix}$ and $\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \epsilon' \begin{pmatrix} b_{12} \\ b_{22} \end{pmatrix}$ and $\begin{pmatrix} a_{11} + a_{12} \\ a_{21} + a_{22} \end{pmatrix} = \epsilon_1 \begin{pmatrix} b_{11} + b_{12} \\ b_{21} + b_{22} \end{pmatrix}$ for some $\epsilon, \epsilon', \epsilon_1 \in \{1, -1\}$. Suppose that $\epsilon\epsilon' = -1$.

Case 1. $\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix}$ and $\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \begin{pmatrix} -b_{12} \\ -b_{22} \end{pmatrix}$. If $\epsilon_1 = 1$, then $a_{11} + a_{12} = b_{11} + b_{12} = b_{11} - b_{12}$ and $a_{21} + a_{22} = b_{21} + b_{22} = b_{21} - b_{22}$, so $b_{12} = b_{22} = 0$. If $\epsilon_1 = -1$, then $a_{11} + a_{12} = -b_{11} - b_{12} = b_{11} - b_{12}$ and $a_{21} + a_{22} = -b_{21} - b_{22} = b_{21} - b_{22}$, so $b_{11} = b_{21} = 0$. Hence, $A = B$.

Case 2. $\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \begin{pmatrix} -b_{11} \\ -b_{21} \end{pmatrix}$ and $\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \begin{pmatrix} b_{12} \\ b_{22} \end{pmatrix}$. If $\epsilon_1 = 1$, then $a_{11} + a_{12} = b_{11} + b_{12} = -b_{11} + b_{12}$ and $a_{21} + a_{22} = b_{21} + b_{22} = -b_{21} + b_{22}$, so $b_{11} = b_{21} = 0$. If $\epsilon_1 = -1$, then $a_{11}+a_{12} = -b_{11}-b_{12} = -b_{11}+b_{12}$ and $a_{21}+a_{22} = -b_{21}-b_{22} = -b_{21}+b_{22}$, so $b_{12} = b_{22} = 0$. Hence, $A = B$.

Step 2. Suppose that the statement is true for $n \in \mathbb{N}$. We show that the statement is also true for $n + 1$.

Suppose that

$$A = \begin{bmatrix} a_{11} & \cdots & a_{12^n} & a_{12^n+1} & \cdots & a_{12^{n+1}} \\ a_{21} & \cdots & a_{22^n} & a_{22^n+1} & \cdots & a_{22^{n+1}} \end{bmatrix}$$

and

$$B = \begin{bmatrix} b_{11} & \cdots & b_{12^n} & b_{12^n+1} & \cdots & b_{12^{n+1}} \\ b_{21} & \cdots & b_{22^n} & b_{22^n+1} & \cdots & b_{22^{n+1}} \end{bmatrix}$$

and

$$A_1 = \begin{pmatrix} a_{11} & \cdots & a_{12^n} \\ a_{21} & \cdots & a_{22^n} \end{pmatrix},\ A_2 = \begin{pmatrix} a_{12^n+1} & \cdots & a_{12^{n+1}} \\ a_{22^n+1} & \cdots & a_{22^{n+1}} \end{pmatrix},$$

$$A_3 = \begin{pmatrix} a_{11} & \cdots & a_{12^{n-1}} & a_{12^n+1} & \cdots & a_{12^n+2^{n-1}} \\ a_{21} & \cdots & a_{22^{n-1}} & a_{22^n+1} & \cdots & a_{22^n+2^{n-1}} \end{pmatrix},$$

$$A_4 = \begin{pmatrix} a_{12^{n-1}+1} & \cdots & a_{12^n} & a_{12^n+2^{n-1}+1} & \cdots & a_{12^{n+1}} \\ a_{22^{n-1}+1} & \cdots & a_{22^n} & a_{22^n+2^{n-1}+1} & \cdots & a_{22^{n+1}} \end{pmatrix}$$

and

$$B_1 = \begin{pmatrix} b_{11} & \cdots & b_{12^n} \\ b_{21} & \cdots & b_{22^n} \end{pmatrix},\ B_2 = \begin{pmatrix} b_{12^n+1} & \cdots & b_{12^{n+1}} \\ b_{22^n+1} & \cdots & b_{22^{n+1}} \end{pmatrix},$$

$$B_3 = \begin{pmatrix} b_{11} & \cdots & b_{12^{n-1}} & b_{12^n+1} & \cdots & b_{12^n+2^{n-1}} \\ b_{21} & \cdots & b_{22^{n-1}} & b_{22^n+1} & \cdots & b_{22^n+2^{n-1}} \end{pmatrix},$$

$$B_4 = \begin{pmatrix} b_{12^{n-1}+1} & \cdots & b_{12^n} & b_{12^n+2^{n-1}+1} & \cdots & b_{12^{n+1}} \\ b_{22^{n-1}+1} & \cdots & b_{22^n} & b_{22^n+2^{n-1}+1} & \cdots & b_{22^{n+1}} \end{pmatrix}.$$

Then $A = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$ and $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$. Notice that

$$[\xi^{n+1}](\{e_1^1, e_2^1, x\}^{n+1}) =$$

$$[\xi^{n+1}](\{e_1^1\} \times \{e_1^1, e_2^1, x\}^n) \coprod [\xi^{n+1}](\{e_2^1\} \times \{e_1^1, e_2^1, x\}^n) \coprod [\xi^{n+1}](\{x\} \times \{e_1^1, e_2^1, x\}^n)$$

and $[\xi^{n+1}](\{e_1^1\} \times \{e_1^1, e_2^1, x\}^n)$, $[\xi^{n+1}](\{e_2^1\} \times \{e_1^1, e_2^1, x\}^n)$, $[\xi^{n+1}](\{x\} \times \{e_1^1, e_2^1, x\}^n)$ have exactly 3^n elements, respectively.

Since $AX = BX$ for each $X \in [\xi^{n+1}](\{e_1^1\} \times \{e_1^1, e_2^1, x\}^n)$, $[A_1]X = [B_1]X$ for each $X \in [\xi^n](\{e_1^1, e_2^1, x\}^n)$.

Similarly, since $AX = BX$ for each $X \in [\xi^{n+1}](\{e_2^1\} \times \{e_1^1, e_2^1, x\}^n)$, $[A_2]X = [B_2]X$ for each $X \in [\xi^n](\{e_1^1, e_2^1, x\}^n)$.

Also, since $AX = BX$ for each $X \in [\xi^{n+1}](\{e_1^1, e_2^1, x\} \times \{e_1^1\} \times \{e_1^1, e_2^1, x\}^{n-1})$, $[A_3]X = [B_3]X$ for each $X \in [\xi^n](\{e_1^1, e_2^1, x\}^n)$.

Similarly, since $AX = BX$ for each $X \in [\xi^{n+1}](\{e_1^1, e_2^1, x\} \times \{e_2^1\} \times \{e_1^1, e_2^1, x\}^{n-1})$, $[A_4]X = [B_4]X$ for each $X \in [\xi^n](\{e_1^1, e_2^1, x\}^n)$.

By induction hypothesis, we have

$$[A_1] = [B_1],\ [A_2] = [B_2],\ [A_3] = [B_3],\ [A_4] = [B_4].$$

Hence, $A_1 = \epsilon B_1$ and $A_2 = \epsilon' B_2$ for some $\epsilon, \epsilon' \in \{1, -1\}$. Now, we claim that, if $\epsilon\epsilon' = -1$, then B_1 or B_2 is the 2×2^n zero matrix.

Suppose that $\epsilon\epsilon' = -1$. Without loss of generality, we may assume that

$$A_1 = B_1 \text{ and } A_2 = -B_2.$$

Suppose that B_1 is not the 2×2^n zero matrix. Then there is $i \in \{1, \ldots, 2^n\}$ such that

$$\begin{pmatrix} b_{1i} \\ b_{2i} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Case 1. If $1 \leq i \leq 2^{n-1}$, then $\begin{pmatrix} b_{12^n+1} & \cdots & b_{12^n+2^{n-1}} \\ b_{22^n+1} & \cdots & b_{22^n+2^{n-1}} \end{pmatrix}$ is the $2 \times 2^{n-1}$ zero matrix since $[A_3] = [B_3]$. We claim that $\begin{pmatrix} b_{12^n+2^{n-1}+1} & \cdots & b_{12^{n+1}} \\ b_{22^n+2^{n-1}+1} & \cdots & b_{22^{n+1}} \end{pmatrix}$ is also the $2 \times 2^{n-1}$ zero matrix.

Suppose that $\begin{pmatrix} b_{12^n+2^{n-1}+1} & \cdots & b_{12^{n+1}} \\ b_{22^n+2^{n-1}+1} & \cdots & b_{22^{n+1}} \end{pmatrix}$ is not the $2 \times 2^{n-1}$ zero matrix. Then there is $j \in \{2^n + 2^{n-1} + 1, \ldots, 2^{n+1}\}$ such that $\begin{pmatrix} b_{1j} \\ b_{2j} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Since $[A_4] = [B_4]$, $\begin{pmatrix} b_{12^{n-1}+1} & \cdots & b_{12^n} \\ b_{22^{n-1}+1} & \cdots & b_{22^n} \end{pmatrix}$ is the $2 \times 2^{n-1}$ zero matrix. In this case, the fact that $AX = BX$ for each $X \in [\xi^{n+1}](\{e_1^1, e_2^1, x\}^{n+1})$ implies

$$\begin{bmatrix} a_{11} & \cdots & a_{12^{n-1}} & a_{12^n+2^{n-1}+1} & \cdots & a_{12^{n+1}} \\ a_{21} & \cdots & a_{22^{n-1}} & a_{22^n+2^{n-1}+1} & \cdots & a_{22^{n+1}} \end{bmatrix} X$$

$$= \begin{bmatrix} b_{11} & \cdots & b_{12^{n-1}} & b_{12^n+2^{n-1}+1} & \cdots & b_{12^{n+1}} \\ b_{21} & \cdots & b_{22^{n-1}} & b_{22^n+2^{n-1}+1} & \cdots & b_{22^{n+1}} \end{bmatrix} X$$

for each $X \in [\xi^n](\{e_1^1, e_2^1, x\}^n)$. Hence, by induction hypothesis, we have

$$\begin{bmatrix} a_{11} & \cdots & a_{12^{n-1}} & a_{12^n+2^{n-1}+1} & \cdots & a_{12^{n+1}} \\ a_{21} & \cdots & a_{22^{n-1}} & a_{22^n+2^{n-1}+1} & \cdots & a_{22^{n+1}} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11} & \cdots & b_{12^{n-1}} & b_{12^n+2^{n-1}+1} & \cdots & b_{12^{n+1}} \\ b_{21} & \cdots & b_{22^{n-1}} & b_{22^n+2^{n-1}+1} & \cdots & b_{22^{n+1}} \end{bmatrix}.$$

Since $A_1 = B_1$ and $A_2 = -B_2$,

$$\begin{bmatrix} b_{11} & \cdots & b_{12^{n-1}} & -b_{12^n+2^{n-1}+1} & \cdots & -b_{12^{n+1}} \\ b_{21} & \cdots & b_{22^{n-1}} & -b_{22^n+2^{n-1}+1} & \cdots & -b_{22^{n+1}} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11} & \cdots & b_{12^{n-1}} & b_{12^n+2^{n-1}+1} & \cdots & b_{12^{n+1}} \\ b_{21} & \cdots & b_{22^{n-1}} & b_{22^n+2^{n-1}+1} & \cdots & b_{22^{n+1}} \end{bmatrix}.$$

Since $\begin{pmatrix} b_{1i} \\ b_{2i} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} b_{12^n+2^{n-1}+1} & \cdots & b_{12^{n+1}} \\ b_{22^n+2^{n-1}+1} & \cdots & b_{22^{n+1}} \end{pmatrix}$ is the $2 \times 2^{n-1}$ zero matrix. This is a contradiction. Therefore, B_2 is the 2×2^n zero matrix.

Similarly, we show the other case.

Case 2. If $2^{n-1} + 1 \leq i \leq 2^n$, then $\begin{pmatrix} b_{12^n+2^{n-1}+1} & \cdots & b_{12^{n+1}} \\ b_{22^n+2^{n-1}+1} & \cdots & b_{22^{n+1}} \end{pmatrix}$ is the $2 \times 2^{n-1}$ zero matrix since $[A_4] = [B_4]$. We claim that $\begin{pmatrix} b_{12^n+1} & \cdots & b_{12^n+2^{n-1}} \\ b_{22^n+1} & \cdots & b_{22^n+2^{n-1}} \end{pmatrix}$ is also the $2 \times 2^{n-1}$ zero matrix.

Suppose that $\begin{pmatrix} b_{12^n+1} & \cdots & b_{12^n+2^{n-1}} \\ b_{22^n+1} & \cdots & b_{22^n+2^{n-1}} \end{pmatrix}$ is not the $2 \times 2^{n-1}$ zero matrix. Then there is $j \in \{2^n + 1, \ldots, 2^n + 2^{n-1}\}$ such that $\begin{pmatrix} b_{1j} \\ b_{2j} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Since $[A_3]=[B_3]$, $\begin{pmatrix} b_{11} & \cdots & b_{12^{n-1}} \\ b_{21} & \cdots & b_{22^{n-1}} \end{pmatrix}$ is the $2\times 2^{n-1}$ zero matrix. In this case, the fact that $AX=BX$ for each $X\in[\xi^{n+1}](\{e_1^1,e_2^1,x\}^{n+1})$ implies

$$\begin{bmatrix} a_{12^{n-1}+1} & \cdots & a_{12^n} & a_{12^n+1} & \cdots & a_{12^n+2^{n-1}} \\ a_{22^{n-1}+1} & \cdots & a_{22^n} & a_{22^n+1} & \cdots & a_{22^n+2^{n-1}} \end{bmatrix} X$$

$$= \begin{bmatrix} b_{12^{n-1}+1} & \cdots & b_{12^n} & b_{12^n+1} & \cdots & b_{12^n+2^{n-1}} \\ b_{22^{n-1}+1} & \cdots & b_{22^n} & b_{22^n+1} & \cdots & b_{22^n+2^{n-1}} \end{bmatrix} X$$

for each $X\in[\xi^n](\{e_1^1,e_2^1,x\}^n)$. Hence, by induction hypothesis and $A_1=B_1$ and $A_2=-B_2$, we have

$$\begin{bmatrix} b_{12^{n-1}+1} & \cdots & b_{12^n} & -b_{12^n+1} & \cdots & -b_{12^n+2^{n-1}} \\ b_{22^{n-1}+1} & \cdots & b_{22^n} & -b_{22^n+1} & \cdots & -b_{22^n+2^{n-1}} \end{bmatrix}$$

$$= \begin{bmatrix} b_{12^{n-1}+1} & \cdots & b_{12^n} & b_{12^n+1} & \cdots & b_{12^n+2^{n-1}} \\ b_{22^{n-1}+1} & \cdots & b_{22^n} & b_{22^n+1} & \cdots & b_{22^n+2^{n-1}} \end{bmatrix}.$$

Since $\begin{pmatrix} b_{1i} \\ b_{2i} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} b_{12^n+1} & \cdots & b_{12^n+2^{n-1}} \\ b_{22^n+1} & \cdots & b_{22^n+2^{n-1}} \end{pmatrix}$ is the $2\times 2^{n-1}$ zero matrix. This is a contradiction. Therefore, B_2 is the 2×2^n zero matrix.

Hence, for each case, we have $A=\begin{bmatrix} A_1 & A_2 \end{bmatrix}=\begin{bmatrix} B_1 & B_2 \end{bmatrix}=B$. This proves the lemma. $\square$

Remark that the invariant of an n-punctured ball tangle is a 2×2^n 'projective matrix' which means a matrix in $PM_{2\times 2^n}(\mathbb{Z})$. To prove Theorem 3.2, we will show that the projective matrices send each of all possible 'projective column vectors' coming from ball tangle invariants the same value. Fortunately, there are ball tangle diagrams whose invariants are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, respectively (See Figure 8).

From this fact, we can say that n-punctured ball tangles have the same invariant if they are the same function on $\boldsymbol{BT}^n$.

Theorem 3.2. *Let $n\in\mathbb{N}$, and let $k_1,\dots,k_n\in\mathbb{N}\cup\{0\}$, and let $T^n,T^{k_1(1)},\dots,T^{k_n(n)}$ be $n,k_1,\dots,k_n$-punctured ball tangle diagrams, respectively. Then*

if $F^{k_1}(T^{k_1(1)})=\begin{bmatrix} b^1_{11} & \cdots & b^1_{12^{k_1}} \\ b^1_{21} & \cdots & b^1_{22^{k_1}} \end{bmatrix},\dots,F^{k_n}(T^{k_n(n)})=\begin{bmatrix} b^n_{11} & \cdots & b^n_{12^{k_n}} \\ b^n_{21} & \cdots & b^n_{22^{k_n}} \end{bmatrix}$, *then*

$F^{k_1+\cdots+k_n}(T^n(T^{k_1(1)},\dots,T^{k_n(n)}))=F^n(T^n)[\eta^n](F^{k_1}(T^{k_1(1)}),\dots,F^{k_n}(T^{k_n(n)}))$,

where $[\eta^n](F^{k_1}(T^{k_1(1)}),\dots,F^{k_n}(T^{k_n(n)}))$

$$
= \begin{bmatrix}
\xi^{n,2^{k_1},\ldots,2^{k_n}}((b^{1}_{\alpha^{n}_{11}1},\ldots,b^{1}_{\alpha^{n}_{11}2^{k_1}}),\ldots,(b^{n}_{\alpha^{n}_{1n}1},\ldots,b^{n}_{\alpha^{n}_{1n}2^{k_n}})) \\
\xi^{n,2^{k_1},\ldots,2^{k_n}}((b^{1}_{\alpha^{n}_{21}1},\ldots,b^{1}_{\alpha^{n}_{21}2^{k_1}}),\ldots,(b^{n}_{\alpha^{n}_{2n}1},\ldots,b^{n}_{\alpha^{n}_{2n}2^{k_n}})) \\
\cdot \\
\cdot \\
\cdot \\
\xi^{n,2^{k_1},\ldots,2^{k_n}}((b^{1}_{\alpha^{n}_{2^{n}1}1},\ldots,b^{1}_{\alpha^{n}_{2^{n}1}2^{k_1}}),\ldots,(b^{n}_{\alpha^{n}_{2^{n}n}1},\ldots,b^{n}_{\alpha^{n}_{2^{n}n}2^{k_n}}))
\end{bmatrix}
$$

$$
= \begin{bmatrix}
\prod_{j=1}^{n} b^{j}_{\alpha^{n}_{1j}\alpha^{n,2^{k_1},\ldots,2^{k_n}}_{1j}} & \prod_{j=1}^{n} b^{j}_{\alpha^{n}_{1j}\alpha^{n,2^{k_1},\ldots,2^{k_n}}_{2j}} & \cdots & \prod_{j=1}^{n} b^{j}_{\alpha^{n}_{1j}\alpha^{n,2^{k_1},\ldots,2^{k_n}}_{2^{k_1+\cdots+k_n}j}} \\
\prod_{j=1}^{n} b^{j}_{\alpha^{n}_{2j}\alpha^{n,2^{k_1},\ldots,2^{k_n}}_{1j}} & \prod_{j=1}^{n} b^{j}_{\alpha^{n}_{2j}\alpha^{n,2^{k_1},\ldots,2^{k_n}}_{2j}} & \cdots & \prod_{j=1}^{n} b^{j}_{\alpha^{n}_{2j}\alpha^{n,2^{k_1},\ldots,2^{k_n}}_{2^{k_1+\cdots+k_n}j}} \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\prod_{j=1}^{n} b^{j}_{\alpha^{n}_{2^{n}j}\alpha^{n,2^{k_1},\ldots,2^{k_n}}_{1j}} & \prod_{j=1}^{n} b^{j}_{\alpha^{n}_{2^{n}j}\alpha^{n,2^{k_1},\ldots,2^{k_n}}_{2j}} & \cdots & \prod_{j=1}^{n} b^{j}_{\alpha^{n}_{2^{n}j}\alpha^{n,2^{k_1},\ldots,2^{k_n}}_{2^{k_1+\cdots+k_n}j}}
\end{bmatrix}.
$$

Proof. Without loss of generality, we may assume that $k_1,\ldots,k_n \in \mathbb{N}$.

Let $T = T^n(T^{k_1(1)},\ldots,T^{k_n(n)})$, and let $B^{(11)},\ldots,B^{(1k_1)},\ldots\ldots,B^{(n1)},\ldots,B^{(nk_n)} \in \boldsymbol{BT}$ with

$$
F^0(B^{(11)}) = \begin{bmatrix} v_1^{11} \\ v_2^{11} \end{bmatrix},\ldots,F^0(B^{(1k_1)}) = \begin{bmatrix} v_1^{1k_1} \\ v_2^{1k_1} \end{bmatrix},
$$

$$
\ldots\ldots,
$$

$$
F^0(B^{(n1)}) = \begin{bmatrix} v_1^{n1} \\ v_2^{n1} \end{bmatrix},\ldots,F^0(B^{(nk_n)}) = \begin{bmatrix} v_1^{nk_n} \\ v_2^{nk_n} \end{bmatrix}.
$$

Then $T(B^{(11)},\ldots,B^{(1k_1)},\ldots\ldots,B^{(n1)},\ldots,B^{(nk_n)})$
$= T^n(T^{k_1(1)}(B^{(11)},\ldots,B^{(1k_1)}),\ldots,T^{k_n(n)}(B^{(n1)},\ldots,B^{(nk_n)}))$ and
$F^0(T(B^{(11)},\ldots,B^{(1k_1)},\ldots\ldots,B^{(n1)},\ldots,B^{(nk_n)}))$
$= F^0(T^n(T^{k_1(1)}(B^{(11)},\ldots,B^{(1k_1)}),\ldots,T^{k_n(n)}(B^{(n1)},\ldots,B^{(nk_n)})))$
$= F^n(T^n)[\xi^n](F^0(T^{k_1(1)}(B^{(11)},\ldots,B^{(1k_1)})),\ldots,F^0(T^{k_n(n)}(B^{(n1)},\ldots,B^{(nk_n)})))$
$= F^n(T^n)[\xi^n](F^{k_1}(T^{k_1(1)})[\xi^{k_1}](F^0(B^{(11)}),\ldots,F^0(B^{(1k_1)})),\ldots,F^{k_n}(T^{k_n(n)})[\xi^{k_n}]$
$(F^0(B^{(n1)}),\ldots,F^0(B^{(nk_n)})))) =$

$$
F^n(T^n)[\xi^n](\begin{bmatrix} b^1_{11} & \cdots & b^1_{12^{k_1}} \\ b^1_{21} & \cdots & b^1_{22^{k_1}} \end{bmatrix} \begin{bmatrix} \prod_{j=1}^{k_1} v^{1j}_{\alpha^{k_1}_{1j}} \\ \cdot \\ \cdot \\ \cdot \\ \prod_{j=1}^{k_1} v^{1j}_{\alpha^{k_1}_{2^{k_1}j}} \end{bmatrix},\ldots,\begin{bmatrix} b^n_{11} & \cdots & b^n_{12^{k_n}} \\ b^n_{21} & \cdots & b^n_{22^{k_n}} \end{bmatrix} \begin{bmatrix} \prod_{j=1}^{k_n} v^{nj}_{\alpha^{k_n}_{1j}} \\ \cdot \\ \cdot \\ \cdot \\ \prod_{j=1}^{k_n} v^{nj}_{\alpha^{k_n}_{2^{k_n}j}} \end{bmatrix})
$$

$$= F^n(T^n)[\xi^n]\left(\begin{bmatrix} b^1_{11}\prod_{j=1}^{k_1} v^{1j}_{\alpha^{k_1}_{1j}} + \cdots + b^1_{12^{k_1}}\prod_{j=1}^{k_1} v^{1j}_{\alpha^{k_1}_{2^{k_1}j}} \\ b^1_{21}\prod_{j=1}^{k_1} v^{1j}_{\alpha^{k_1}_{1j}} + \cdots + b^1_{22^{k_1}}\prod_{j=1}^{k_1} v^{1j}_{\alpha^{k_1}_{2^{k_1}j}} \end{bmatrix}, \ldots,\right.$$

$$\left.\begin{bmatrix} b^n_{11}\prod_{j=1}^{k_n} v^{nj}_{\alpha^{k_n}_{1j}} + \cdots + b^n_{12^{k_n}}\prod_{j=1}^{k_n} v^{nj}_{\alpha^{k_n}_{2^{k_n}j}} \\ b^n_{21}\prod_{j=1}^{k_n} v^{nj}_{\alpha^{k_n}_{1j}} + \cdots + b^n_{22^{k_n}}\prod_{j=1}^{k_n} v^{nj}_{\alpha^{k_n}_{2^{k_n}j}} \end{bmatrix}\right)$$

$$= F^n(T^n)\begin{bmatrix} \xi^{n,2^{k_1},\ldots,2^{k_n}}((b^1_{\alpha^n_{11}1},\ldots,b^1_{\alpha^n_{11}2^{k_1}}),\ldots,(b^n_{\alpha^n_{1n}1},\ldots,b^n_{\alpha^n_{1n}2^{k_n}})) \\ \xi^{n,2^{k_1},\ldots,2^{k_n}}((b^1_{\alpha^n_{21}1},\ldots,b^1_{\alpha^n_{21}2^{k_1}}),\ldots,(b^n_{\alpha^n_{2n}1},\ldots,b^n_{\alpha^n_{2n}2^{k_n}})) \\ \cdot \\ \cdot \\ \cdot \\ \xi^{n,2^{k_1},\ldots,2^{k_n}}((b^1_{\alpha^n_{2^n1}1},\ldots,b^1_{\alpha^n_{2^n1}2^{k_1}}),\ldots,(b^n_{\alpha^n_{2^nn}1},\ldots,b^n_{\alpha^n_{2^nn}2^{k_n}})) \end{bmatrix}\times$$

$$\left[\xi^{n,2^{k_1},\ldots,2^{k_n}}((\prod_{j=1}^{k_1} v^{1j}_{\alpha^{k_1}_{1j}},\ldots,\prod_{j=1}^{k_1} v^{1j}_{\alpha^{k_1}_{2^{k_1}j}}),\ldots,(\prod_{j=1}^{k_n} v^{nj}_{\alpha^{k_n}_{1j}},\ldots,\prod_{j=1}^{k_n} v^{nj}_{\alpha^{k_n}_{2^{k_n}j}}))\right]^\dagger$$

$$= F^n(T^n)\begin{bmatrix} \xi^{n,2^{k_1},\ldots,2^{k_n}}((b^1_{\alpha^n_{11}1},\ldots,b^1_{\alpha^n_{11}2^{k_1}}),\ldots,(b^n_{\alpha^n_{1n}1},\ldots,b^n_{\alpha^n_{1n}2^{k_n}})) \\ \xi^{n,2^{k_1},\ldots,2^{k_n}}((b^1_{\alpha^n_{21}1},\ldots,b^1_{\alpha^n_{21}2^{k_1}}),\ldots,(b^n_{\alpha^n_{2n}1},\ldots,b^n_{\alpha^n_{2n}2^{k_n}})) \\ \cdot \\ \cdot \\ \cdot \\ \xi^{n,2^{k_1},\ldots,2^{k_n}}((b^1_{\alpha^n_{2^n1}1},\ldots,b^1_{\alpha^n_{2^n1}2^{k_1}}),\ldots,(b^n_{\alpha^n_{2^nn}1},\ldots,b^n_{\alpha^n_{2^nn}2^{k_n}})) \end{bmatrix}\times$$

$$[\xi^{n,2^{k_1},\ldots,2^{k_n}}]\left(\begin{bmatrix} \prod_{j=1}^{k_1} v^{1j}_{\alpha^{k_1}_{1j}} \\ \cdot \\ \cdot \\ \cdot \\ \prod_{j=1}^{k_1} v^{1j}_{\alpha^{k_1}_{2^{k_1}j}} \end{bmatrix}, \ldots, \begin{bmatrix} \prod_{j=1}^{k_n} v^{nj}_{\alpha^{k_n}_{1j}} \\ \cdot \\ \cdot \\ \cdot \\ \prod_{j=1}^{k_n} v^{nj}_{\alpha^{k_n}_{2^{k_n}j}} \end{bmatrix}\right)$$

$$= F^n(T^n)\begin{bmatrix} \xi^{n,2^{k_1},\ldots,2^{k_n}}((b^1_{\alpha^n_{11}1},\ldots,b^1_{\alpha^n_{11}2^{k_1}}),\ldots,(b^n_{\alpha^n_{1n}1},\ldots,b^n_{\alpha^n_{1n}2^{k_n}})) \\ \xi^{n,2^{k_1},\ldots,2^{k_n}}((b^1_{\alpha^n_{21}1},\ldots,b^1_{\alpha^n_{21}2^{k_1}}),\ldots,(b^n_{\alpha^n_{2n}1},\ldots,b^n_{\alpha^n_{2n}2^{k_n}})) \\ \cdot \\ \cdot \\ \cdot \\ \xi^{n,2^{k_1},\ldots,2^{k_n}}((b^1_{\alpha^n_{2^n1}1},\ldots,b^1_{\alpha^n_{2^n1}2^{k_1}}),\ldots,(b^n_{\alpha^n_{2^nn}1},\ldots,b^n_{\alpha^n_{2^nn}2^{k_n}})) \end{bmatrix}\times$$

$$\begin{bmatrix} \prod_{j=1}^{k_1} v^{1j}_{\alpha^{k_1}_{1j}} \cdots \prod_{j=1}^{k_{n-2}} v^{n-2j}_{\alpha^{k_{n-2}}_{1j}} \prod_{j=1}^{k_{n-1}} v^{n-1j}_{\alpha^{k_{n-1}}_{1j}} \prod_{j=1}^{k_n} v^{nj}_{\alpha^{k_n}_{1j}} \\ \prod_{j=1}^{k_1} v^{1j}_{\alpha^{k_1}_{1j}} \cdots \prod_{j=1}^{k_{n-2}} v^{n-2j}_{\alpha^{k_{n-2}}_{1j}} \prod_{j=1}^{k_{n-1}} v^{n-1j}_{\alpha^{k_{n-1}}_{1j}} \prod_{j=1}^{k_n} v^{nj}_{\alpha^{k_n}_{2j}} \\ \cdot \\ \cdot \\ \cdot \\ \prod_{j=1}^{k_1} v^{1j}_{\alpha^{k_1}_{1j}} \cdots \prod_{j=1}^{k_{n-2}} v^{n-2j}_{\alpha^{k_{n-2}}_{1j}} \prod_{j=1}^{k_{n-1}} v^{n-1j}_{\alpha^{k_{n-1}}_{1j}} \prod_{j=1}^{k_n} v^{nj}_{\alpha^{k_n}_{2^{k_n}j}} \\ \prod_{j=1}^{k_1} v^{1j}_{\alpha^{k_1}_{1j}} \cdots \prod_{j=1}^{k_{n-2}} v^{n-2j}_{\alpha^{k_{n-2}}_{1j}} \prod_{j=1}^{k_{n-1}} v^{n-1j}_{\alpha^{k_{n-1}}_{2j}} \prod_{j=1}^{k_n} v^{nj}_{\alpha^{k_n}_{1j}} \\ \prod_{j=1}^{k_1} v^{1j}_{\alpha^{k_1}_{1j}} \cdots \prod_{j=1}^{k_{n-2}} v^{n-2j}_{\alpha^{k_{n-2}}_{1j}} \prod_{j=1}^{k_{n-1}} v^{n-1j}_{\alpha^{k_{n-1}}_{2j}} \prod_{j=1}^{k_n} v^{nj}_{\alpha^{k_n}_{2j}} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \prod_{j=1}^{k_1} v^{1j}_{\alpha^{k_1}_{2^{k_1}j}} \cdots \prod_{j=1}^{k_{n-2}} v^{n-2j}_{\alpha^{k_{n-2}}_{2^{k_{n-2}}j}} \prod_{j=1}^{k_{n-1}} v^{n-1j}_{\alpha^{k_{n-1}}_{2^{k_{n-1}}j}} \prod_{j=1}^{k_n} v^{nj}_{\alpha^{k_n}_{2^{k_n}j}} \end{bmatrix} =$$

$F^{k_1+\cdots+k_n}(T)[\xi^{k_1+\cdots+k_n}](F^0(B^{(11)}),\ldots,F^0(B^{(1k_1)}),\ldots\ldots,F^0(B^{(n1)}),\ldots,F^0(B^{(nk_n)}))$.

Notice that there are ball tangle diagrams $B^{(1)}, B^{(2)}, B^{(3)}$ such that

$F^0(B^{(1)}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, F^0(B^{(2)}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, F^0(B^{(3)}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, respectively (See Figure 8).

Therefore, by Lemma 3.1,

$F^{k_1+\cdots+k_n}(T^n(T^{k_1(1)},\ldots,T^{k_n(n)}))$

$$= F^n(T^n) \begin{bmatrix} \xi^{n,2^{k_1},\ldots,2^{k_n}}((b^1_{\alpha^n_{11}1},\ldots,b^1_{\alpha^n_{11}2^{k_1}}),\ldots,(b^n_{\alpha^n_{1n}1},\ldots,b^n_{\alpha^n_{1n}2^{k_n}})) \\ \xi^{n,2^{k_1},\ldots,2^{k_n}}((b^1_{\alpha^n_{21}1},\ldots,b^1_{\alpha^n_{21}2^{k_1}}),\ldots,(b^n_{\alpha^n_{2n}1},\ldots,b^n_{\alpha^n_{2n}2^{k_n}})) \\ \cdot \\ \cdot \\ \cdot \\ \xi^{n,2^{k_1},\ldots,2^{k_n}}((b^1_{\alpha^n_{2^n1}1},\ldots,b^1_{\alpha^n_{2^n1}2^{k_1}}),\ldots,(b^n_{\alpha^n_{2^nn}1},\ldots,b^n_{\alpha^n_{2^nn}2^{k_n}})) \end{bmatrix} =$$

$$F^n(T^n)\begin{bmatrix} \prod_{j=1}^n b^j_{\alpha^n_{1j}\alpha^{n,2^{k_1},\dots,2^{k_n}}_{1j}} & \prod_{j=1}^n b^j_{\alpha^n_{1j}\alpha^{n,2^{k_1},\dots,2^{k_n}}_{2j}} & \cdots & \prod_{j=1}^n b^j_{\alpha^n_{1j}\alpha^{n,2^{k_1},\dots,2^{k_n}}_{2^{k_1+\cdots+k_n}j}} \\ \prod_{j=1}^n b^j_{\alpha^n_{2j}\alpha^{n,2^{k_1},\dots,2^{k_n}}_{1j}} & \prod_{j=1}^n b^j_{\alpha^n_{2j}\alpha^{n,2^{k_1},\dots,2^{k_n}}_{2j}} & \cdots & \prod_{j=1}^n b^j_{\alpha^n_{2j}\alpha^{n,2^{k_1},\dots,2^{k_n}}_{2^{k_1+\cdots+k_n}j}} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \prod_{j=1}^n b^j_{\alpha^n_{2^n j}\alpha^{n,2^{k_1},\dots,2^{k_n}}_{1j}} & \prod_{j=1}^n b^j_{\alpha^n_{2^n j}\alpha^{n,2^{k_1},\dots,2^{k_n}}_{2j}} & \cdots & \prod_{j=1}^n b^j_{\alpha^n_{2^n j}\alpha^{n,2^{k_1},\dots,2^{k_n}}_{2^{k_1+\cdots+k_n}j}} \end{bmatrix}$$

$= F^n(T^n)[\eta^n](F^{k_1}(T^{k_1(1)}),\dots,F^{k_n}(T^{k_n(n)}))$.

This proves the theorem. □

Let us give the following example.

Suppose that $T^3, T^{1(1)}, T^{1(2)}, T^{1(3)}$ are $3,1,1,1$-punctured ball tangle diagrams such that

$$F^3(T^3) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} \end{bmatrix},$$

$$F^1(T^{1(1)}) = \begin{bmatrix} b^1_{11} & b^1_{12} \\ b^1_{21} & b^1_{22} \end{bmatrix},\ F^1(T^{1(2)}) = \begin{bmatrix} b^2_{11} & b^2_{12} \\ b^2_{21} & b^2_{22} \end{bmatrix},\ F^1(T^{1(3)}) = \begin{bmatrix} b^3_{11} & b^3_{12} \\ b^3_{21} & b^3_{22} \end{bmatrix},$$

respectively. Then $T^3(T^{1(1)}, T^{1(2)}, T^{1(3)})$ is a 3-punctured ball tangle diagram and

$$F^3(T^3(T^{1(1)}, T^{1(2)}, T^{1(3)})) = F^3(T^3)[\eta^3](F^1(T^{1(1)}), F^1(T^{1(2)}), F^1(T^{1(3)}))$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} \end{bmatrix} \times$$

$$\begin{bmatrix}
b^1_{11}b^2_{11}b^3_{11} & b^1_{11}b^2_{11}b^3_{12} & b^1_{11}b^2_{12}b^3_{11} & b^1_{11}b^2_{12}b^3_{12} & b^1_{12}b^2_{11}b^3_{11} & b^1_{12}b^2_{11}b^3_{12} & b^1_{12}b^2_{12}b^3_{11} & b^1_{12}b^2_{12}b^3_{12} \\
b^1_{11}b^2_{11}b^3_{21} & b^1_{11}b^2_{11}b^3_{22} & b^1_{11}b^2_{12}b^3_{21} & b^1_{11}b^2_{12}b^3_{22} & b^1_{12}b^2_{11}b^3_{21} & b^1_{12}b^2_{11}b^3_{22} & b^1_{12}b^2_{12}b^3_{21} & b^1_{12}b^2_{12}b^3_{22} \\
b^1_{11}b^2_{21}b^3_{11} & b^1_{11}b^2_{21}b^3_{12} & b^1_{11}b^2_{22}b^3_{11} & b^1_{11}b^2_{22}b^3_{12} & b^1_{12}b^2_{21}b^3_{11} & b^1_{12}b^2_{21}b^3_{12} & b^1_{12}b^2_{22}b^3_{11} & b^1_{12}b^2_{22}b^3_{12} \\
b^1_{11}b^2_{21}b^3_{21} & b^1_{11}b^2_{21}b^3_{22} & b^1_{11}b^2_{22}b^3_{21} & b^1_{11}b^2_{22}b^3_{22} & b^1_{12}b^2_{21}b^3_{21} & b^1_{12}b^2_{21}b^3_{22} & b^1_{12}b^2_{22}b^3_{21} & b^1_{12}b^2_{22}b^3_{22} \\
b^1_{21}b^2_{11}b^3_{11} & b^1_{21}b^2_{11}b^3_{12} & b^1_{21}b^2_{12}b^3_{11} & b^1_{21}b^2_{12}b^3_{12} & b^1_{22}b^2_{11}b^3_{11} & b^1_{22}b^2_{11}b^3_{12} & b^1_{22}b^2_{12}b^3_{11} & b^1_{22}b^2_{12}b^3_{12} \\
b^1_{21}b^2_{11}b^3_{21} & b^1_{21}b^2_{11}b^3_{22} & b^1_{21}b^2_{12}b^3_{21} & b^1_{21}b^2_{12}b^3_{22} & b^1_{22}b^2_{11}b^3_{21} & b^1_{22}b^2_{11}b^3_{22} & b^1_{22}b^2_{12}b^3_{21} & b^1_{22}b^2_{12}b^3_{22} \\
b^1_{21}b^2_{21}b^3_{11} & b^1_{21}b^2_{21}b^3_{12} & b^1_{21}b^2_{22}b^3_{11} & b^1_{21}b^2_{22}b^3_{12} & b^1_{22}b^2_{21}b^3_{11} & b^1_{22}b^2_{21}b^3_{12} & b^1_{22}b^2_{22}b^3_{11} & b^1_{22}b^2_{22}b^3_{12} \\
b^1_{21}b^2_{21}b^3_{21} & b^1_{21}b^2_{21}b^3_{22} & b^1_{21}b^2_{22}b^3_{21} & b^1_{21}b^2_{22}b^3_{22} & b^1_{22}b^2_{21}b^3_{21} & b^1_{22}b^2_{21}b^3_{22} & b^1_{22}b^2_{22}b^3_{21} & b^1_{22}b^2_{22}b^3_{22}
\end{bmatrix}.$$

Next, let us consider '(outer) connect sums' of various n-punctured ball tangle diagrams and their invariants. They will be also very useful when we compute invariants of complicated tangles. Given k_1 and k_2-punctured ball tangle diagrams $T^{k_1(1)}$ and $T^{k_2(2)}$, we denote the 'horizontal' and the 'vertical' connect sums of them by $T^{k_1(1)} +_h T^{k_2(2)}$ and $T^{k_1(1)} +_v T^{k_2(2)}$, respectively (See Figure 7).

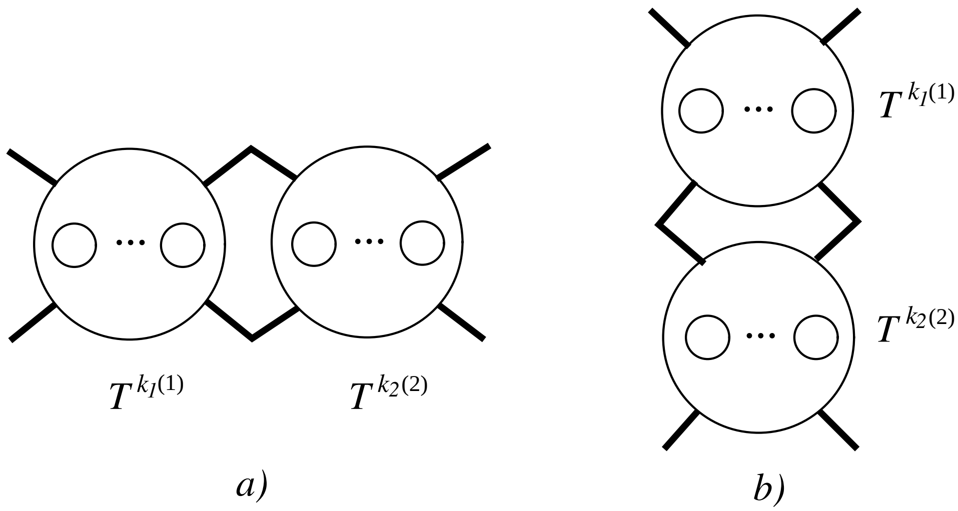


Figure 7. Connect sums of punctured ball tangles.
a) $T^{k_1(1)} +_h T^{k_2(2)}$, b) $T^{k_1(1)} +_v T^{k_2(2)}$.

Theorem 3.3. *Let $k_1, k_2 \in \mathbb{N} \cup \{0\}$, and let $T^{k_1(1)}, T^{k_2(2)}$ be k_1, k_2-punctured ball tangle diagrams, respectively. Then*

if $F^{k_1}(T^{k_1(1)}) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{12^{k_1}} \\ a_{21} & a_{22} & \cdots & a_{22^{k_1}} \end{bmatrix}$ *and* $F^{k_2}(T^{k_2(2)}) = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{12^{k_2}} \\ b_{21} & b_{22} & \cdots & b_{22^{k_2}} \end{bmatrix}$,

then

(1) $F^{k_1+k_2}(T^{k_1(1)} +_h T^{k_2(2)}) = \left[\left(\begin{pmatrix} a_{1i}b_{2j} + a_{2i}b_{1j} \\ a_{2i}b_{2j} \end{pmatrix}_{j=1,\ldots,2^{k_2}} \right)_{i=1,\ldots,2^{k_1}} \right]$,

(2) $F^{k_1+k_2}(T^{k_1(1)} +_v T^{k_2(2)}) = \left[\left(\begin{pmatrix} a_{1i}b_{1j} \\ a_{2i}b_{1j} + a_{1i}b_{2j} \end{pmatrix}_{j=1,\ldots,2^{k_2}} \right)_{i=1,\ldots,2^{k_1}} \right]$.

Proof. We denote $F^0(B^{(1)} +_h B^{(2)})$ by $F^0(B^{(1)}) +_h F^0(B^{(2)})$ and $F^0(B^{(1)} +_v B^{(2)})$ by $F^0(B^{(1)}) +_v F^0(B^{(2)})$ if $B^{(1)}, B^{(2)} \in \boldsymbol{BT}$.

(1) Let $T = T^{k_1(1)} +_h T^{k_2(2)}$, and let $B^{(11)}, \ldots, B^{(1k_1)}, B^{(21)}, \ldots, B^{(2k_2)} \in \boldsymbol{BT}$ with

$$F^0(B^{(11)}) = \begin{bmatrix} v_1^{11} \\ v_2^{11} \end{bmatrix}, \ldots, F^0(B^{(1k_1)}) = \begin{bmatrix} v_1^{1k_1} \\ v_2^{1k_1} \end{bmatrix},$$

$$F^0(B^{(21)}) = \begin{bmatrix} v_1^{21} \\ v_2^{21} \end{bmatrix}, \ldots, F^0(B^{(2k_2)}) = \begin{bmatrix} v_1^{2k_2} \\ v_2^{2k_2} \end{bmatrix}.$$

Then $T(B^{(11)}, \ldots, B^{(1k_1)}, B^{(21)}, \ldots, B^{(2k_2)})$
$= T^{k_1(1)}(B^{(11)}, \ldots, B^{(1k_1)}) +_h T^{k_2(2)}(B^{(21)}, \ldots, B^{(2k_2)})$ and

$F^0(T(B^{(11)},\dots,B^{(1k_1)},B^{(21)},\dots,B^{(2k_2)}))$
$= F^0(T^{k_1(1)}(B^{(11)},\dots,B^{(1k_1)}) +_h T^{k_2(2)}(B^{(21)},\dots,B^{(2k_2)}))$
$= F^0(T^{k_1(1)}(B^{(11)},\dots,B^{(1k_1)})) +_h F^0(T^{k_2(2)}(B^{(21)},\dots,B^{(2k_2)}))$
$= F^{k_1}(T^{k_1(1)})[\xi^{k_1}](F^0(B^{(11)}),\dots,F^0(B^{(1k_1)}))$
$+_h F^{k_2}(T^{k_2(2)})[\xi^{k_2}](F^0(B^{(21)}),\dots,F^0(B^{(2k_2)}))$

$$= \begin{bmatrix} a_{11} & \cdots & a_{12^{k_1}} \\ a_{21} & \cdots & a_{22^{k_1}} \end{bmatrix} \begin{bmatrix} \prod_{j=1}^{k_1} v^{1j}_{\alpha^{k_1}_{1j}} \\ \cdot \\ \cdot \\ \cdot \\ \prod_{j=1}^{k_1} v^{1j}_{\alpha^{k_1}_{2^{k_1}j}} \end{bmatrix} +_h \begin{bmatrix} b_{11} & \cdots & b_{12^{k_2}} \\ b_{21} & \cdots & b_{22^{k_2}} \end{bmatrix} \begin{bmatrix} \prod_{j=1}^{k_2} v^{2j}_{\alpha^{k_2}_{1j}} \\ \cdot \\ \cdot \\ \cdot \\ \prod_{j=1}^{k_2} v^{2j}_{\alpha^{k_2}_{2^{k_2}j}} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11}\prod_{j=1}^{k_1} v^{1j}_{\alpha^{k_1}_{1j}} + \cdots + a_{12^{k_1}}\prod_{j=1}^{k_1} v^{1j}_{\alpha^{k_1}_{2^{k_1}j}} \\ a_{21}\prod_{j=1}^{k_1} v^{1j}_{\alpha^{k_1}_{1j}} + \cdots + a_{22^{k_1}}\prod_{j=1}^{k_1} v^{1j}_{\alpha^{k_1}_{2^{k_1}j}} \end{bmatrix} +_h \begin{bmatrix} b_{11}\prod_{j=1}^{k_2} v^{2j}_{\alpha^{k_2}_{1j}} + \cdots + b_{12^{k_2}}\prod_{j=1}^{k_2} v^{2j}_{\alpha^{k_2}_{2^{k_2}j}} \\ b_{21}\prod_{j=1}^{k_2} v^{2j}_{\alpha^{k_2}_{1j}} + \cdots + b_{22^{k_2}}\prod_{j=1}^{k_2} v^{2j}_{\alpha^{k_2}_{2^{k_2}j}} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{21} + a_{21}b_{11} & a_{21}b_{21} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ a_{11}b_{22^{k_2}} + a_{21}b_{12^{k_2}} & a_{21}b_{22^{k_2}} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ a_{12^{k_1}}b_{21} + a_{22^{k_1}}b_{11} & a_{22^{k_1}}b_{21} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ a_{12^{k_1}}b_{22^{k_2}} + a_{22^{k_1}}b_{12^{k_2}} & a_{22^{k_1}}b_{22^{k_2}} \end{bmatrix}^{\dagger} \begin{bmatrix} \prod_{j=1}^{k_1} v^{1j}_{\alpha^{k_1}_{1j}} \prod_{j=1}^{k_2} v^{2j}_{\alpha^{k_2}_{1j}} \\ \cdot \\ \cdot \\ \cdot \\ \prod_{j=1}^{k_1} v^{1j}_{\alpha^{k_1}_{1j}} \prod_{j=1}^{k_2} v^{2j}_{\alpha^{k_2}_{2^{k_2}j}} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \prod_{j=1}^{k_1} v^{1j}_{\alpha^{k_1}_{1j}} \prod_{j=1}^{k_2} v^{2j}_{\alpha^{k_2}_{1j}} \\ \cdot \\ \cdot \\ \cdot \\ \prod_{j=1}^{k_1} v^{1j}_{\alpha^{k_1}_{2^{k_1}j}} \prod_{j=1}^{k_2} v^{2j}_{\alpha^{k_2}_{2^{k_2}j}} \end{bmatrix}$$

$= F^{k_1+k_2}(T)[\xi^{k_1+k_2}](F^0(B^{(11)}),\dots,F^0(B^{(1k_1)}),F^0(B^{(21)}),\dots,F^0(B^{(2k_2)}))$.

Notice that there are ball tangle diagrams $B^{(1)}, B^{(2)}, B^{(3)}$ such that

$F^0(B^{(1)}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $F^0(B^{(2)}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $F^0(B^{(3)}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, respectively (See Figure 8).

Therefore, by Lemma 3.1,

$$F^{k_1+k_2}(T^{k_1(1)} +_h T^{k_2(2)}) = \left[\left(\begin{pmatrix} a_{1i}b_{2j} + a_{2i}b_{1j} \\ a_{2i}b_{2j} \end{pmatrix}_{j=1,\ldots,2^{k_2}}\right)_{i=1,\ldots,2^{k_1}}\right].$$

(2) Similarly, we can show that

$$F^{k_1+k_2}(T^{k_1(1)} +_v T^{k_2(2)}) = \left[\left(\begin{pmatrix} a_{1i}b_{1j} \\ a_{2i}b_{1j} + a_{1i}b_{2j} \end{pmatrix}_{j=1,\ldots,2^{k_2}}\right)_{i=1,\ldots,2^{k_1}}\right].$$

This proves the theorem. □

Notice that each of the horizontal connect sum and the vertical connect sum of punctured ball tangles is associative but not commutative. However, their invariants are not changed.

Corollary 3.4. *Let $k_1, k_2 \in \mathbb{N} \cup \{0\}$, and let $T^{k_1(1)}, T^{k_2(2)}$ be k_1, k_2-punctured ball tangle diagrams, respectively. Then*
(1) $F^{k_1+k_2}(T^{k_1(1)} +_h T^{k_2(2)}) = F^{k_2+k_1}(T^{k_2(2)} +_h T^{k_1(1)})$,
(2) $F^{k_1+k_2}(T^{k_1(1)} +_v T^{k_2(2)}) = F^{k_2+k_1}(T^{k_2(2)} +_v T^{k_1(1)})$.

From now on, we denote simply by F and f for F^1 and F^0, respectively. The following corollaries of our main theorems are for the invariants of ball tangles and spherical tangles.

Corollary 3.5. *If $A, B \in PM_{2\times 2}(\mathbb{Z})$ and*
$A\begin{bmatrix}1\\0\end{bmatrix} = B\begin{bmatrix}1\\0\end{bmatrix}, A\begin{bmatrix}0\\1\end{bmatrix} = B\begin{bmatrix}0\\1\end{bmatrix}, A\begin{bmatrix}1\\1\end{bmatrix} = B\begin{bmatrix}1\\1\end{bmatrix}$*, then $A = B$.*

Corollary 3.6. *If $S^{(1)}, S^{(2)} \in \boldsymbol{ST}$, then $F(S^{(2)}(S^{(1)})) = F(S^{(2)})F(S^{(1)})$.*

Corollary 3.7. *If $B^{(1)}, B^{(2)} \in \boldsymbol{BT}$ with $f(B^{(1)}) = \begin{bmatrix}p\\q\end{bmatrix}$ and $f(B^{(2)}) = \begin{bmatrix}r\\s\end{bmatrix}$, then*

(1) $f(B^{(1)} +_h B^{(2)}) = \begin{bmatrix}ps+qr\\qs\end{bmatrix}$ (Krebes [5]), (2) $f(B^{(1)} +_v B^{(2)}) = \begin{bmatrix}pr\\qr+ps\end{bmatrix}$.

Corollary 3.8. *If $B \in \boldsymbol{BT}$ with $f(B) = \begin{bmatrix}p\\q\end{bmatrix}$ and $S \in \boldsymbol{ST}$ with $F(S) = \begin{bmatrix}\alpha & \gamma\\ \beta & \delta\end{bmatrix}$, then*

(1) $F(B +_h S) = \begin{bmatrix}p\beta + q\alpha & p\delta + q\gamma\\ q\beta & q\delta\end{bmatrix}$, (2) $F(B +_v S) = \begin{bmatrix}p\alpha & p\gamma\\ q\alpha + p\beta & q\gamma + p\delta\end{bmatrix}$.

A connect sum of two spherical tangles is a 2-punctured ball tangle, so it has a 2×2^2 matrix in $PM_{2\times 2^2}(\mathbb{Z})$. As a corollary of Theorem 3.3, we give one more statement as follows.

Corollary 3.9. *If* $S^{(1)}, S^{(2)} \in \boldsymbol{ST}$ *with* $F(S^{(1)}) = \begin{bmatrix} p & r \\ q & s \end{bmatrix}$ *and* $F(S^{(2)}) = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$, *then*

(1) $F^2(S^{(1)} +_h S^{(2)}) = \begin{bmatrix} p\beta + q\alpha & p\delta + q\gamma & r\beta + s\alpha & r\delta + s\gamma \\ q\beta & q\delta & s\beta & s\delta \end{bmatrix}$,

(2) $F^2(S^{(1)} +_v S^{(2)}) = \begin{bmatrix} p\alpha & p\gamma & r\alpha & r\gamma \\ q\alpha + p\beta & q\gamma + p\delta & s\alpha + r\beta & s\gamma + r\delta \end{bmatrix}$.

Let us calculate the invariant for each of the ball tangles and the spherical tangles in Figure 8.

1. The fundamental ball tangles $\boldsymbol{a}$ and $\boldsymbol{b}$ have invariants $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, respectively.
2. The ball tangle $\boldsymbol{c}$ has invariant $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
3. The spherical tangle $\boldsymbol{d}$ is $\boldsymbol{I}$ and has invariant $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
4. The spherical tangle $\boldsymbol{e}$ has invariant $\begin{bmatrix} 1 \\ 1 \end{bmatrix} +_v \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.
5. The spherical tangle $\boldsymbol{f}$ has invariant $\begin{bmatrix} 1 \\ 0 \end{bmatrix} +_h \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

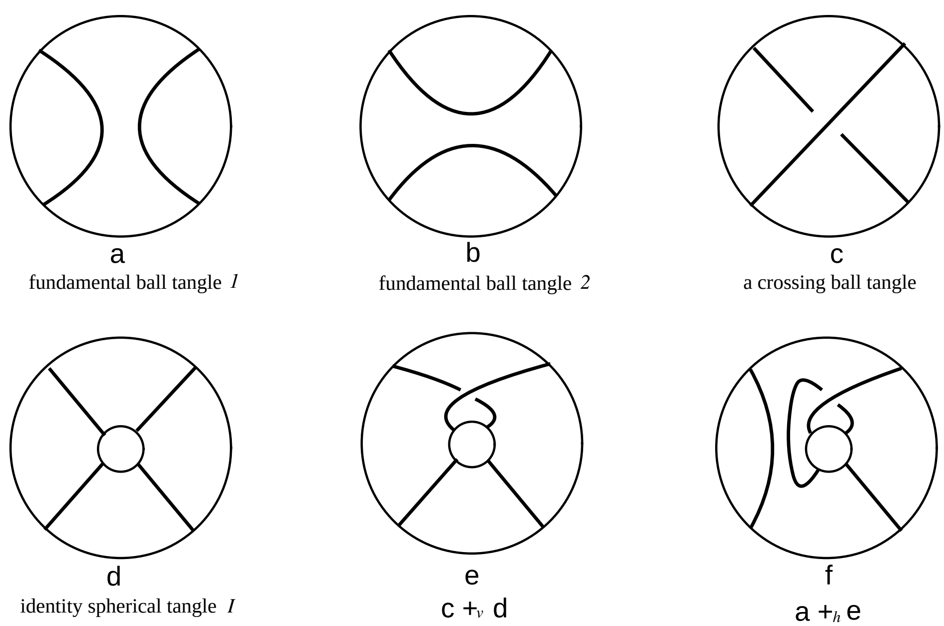


Figure 8. Ball tangle diagrams and spherical tangle diagrams.

When we denote the statement that n-punctured ball tangles $T^{n(1)}$ and $T^{n(2)}$ induces the same function from $\boldsymbol{BT}^n$ to $\boldsymbol{BT}$ by $T^{n(1)} \simeq T^{n(2)}$ and $F^n(T^{n(1)}) = F^n(T^{n(2)})$ by $T^{n(1)} \sim T^{n(2)}$, $\simeq$ and $\sim$ are clearly equivalence relations on $\boldsymbol{nPBT}$ and we have

$$T^{n(1)} \cong T^{n(2)} \Longrightarrow T^{n(1)} \simeq T^{n(2)} \Longrightarrow T^{n(1)} \sim T^{n(2)}.$$

The first implication comes from the definition of $\cong$ and the second implication is proved by Theorem 2.14 and Lemma 3.1 immediately.

Notice that neither the converse of the first implication nor that of the second implication is true (See Figure 9). In particular, the spherical tangles C and D in Figure 9 have the matrix $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ as invariant. For another nonzero matrix invariant, we can take the spherical tangle A in Figure 9 and a spherical tangle B' obtained from a single twist of the hole of A. We easily know that A and B' are different functions. However, A and B' have the same invariant. By these reasons, we may consider the equivalence relation $\simeq$ instead of $\cong$ for our n-punctured ball tangle invariant.

This aspect is quite similar to that in Algebraic Topology in the sense as follows:

If X and Y are pathconnected topological spaces, then

$$X \cong Y \Longrightarrow X \simeq Y \Longrightarrow X \sim Y,$$

where $X \cong Y$, $X \simeq Y$, and $X \sim Y$ mean the statements that X and Y are topologically equivalent, X and Y are homotopically equivalent, and $\pi_1(X)$ and $\pi_1(Y)$ are isomorphic, respectively.

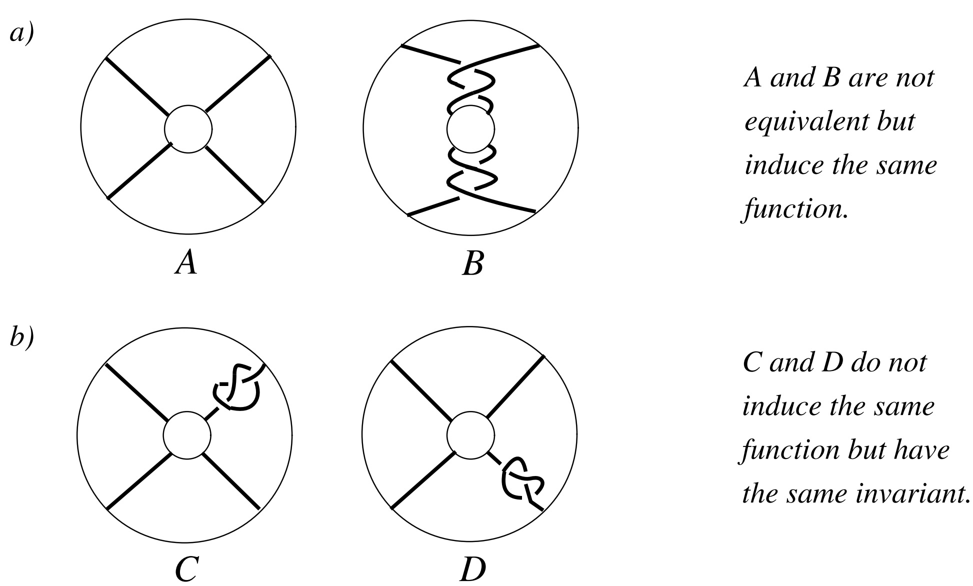


Figure 9. Tangles and functions.

Lemma 3.10 (J.-W. Chung and X.-S. Lin [4])**.** *Let $\boldsymbol{J}$ be the spherical tangle shown in Figure 10. Let p_1, p_2, p_3, p_4 be the number of half twists inside of the balls marked by 1,2,3,4, respectively. Then*

$$F(\boldsymbol{J}) = \begin{bmatrix} p_1p_2p_3 + p_1p_2p_4 + p_1p_3p_4 + p_2p_3p_4 & -p_1p_3 - p_1p_4 - p_2p_3 - p_2p_4 \\ p_1p_2 + p_1p_4 + p_3p_2 + p_3p_4 & -p_1 - p_2 - p_3 - p_4 \end{bmatrix}.$$

Therefore,

$$\det F(\boldsymbol{J}) = (p_1p_4 - p_2p_3)^2.$$

This is by a direct calculation.

Now, let us indicate a direct way to compute the invariant $F(\boldsymbol{J})$ of the spherical tangle $\boldsymbol{J}$ in Figure 10, in the special case of $p_1 = p_2 = p_4 = -4$ and $p_3 = 2$. Check with the formula in Lemma 3.10. Suppose that $T^5, B^{(1)}, B^{(2)}, B^{(3)}, B^{(4)}$ are the $5, 0, 0, 0, 0$-punctured ball tangle diagrams in Figure 10, respectively. Then

$$\boldsymbol{J} = T^5(B^{(1)}, B^{(2)}, B^{(3)}, B^{(4)}, \boldsymbol{I}).$$

By Theorem 3.2, $F(\boldsymbol{J}) = F^5(T^5)[\eta^5](f(B^{(1)}), f(B^{(2)}), f(B^{(3)}), f(B^{(4)}), F(\boldsymbol{I}))$.

We have $f(B^{(1)}) = f(B^{(2)}) = f(B^{(4)}) = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$ and $f(B^{(3)}) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

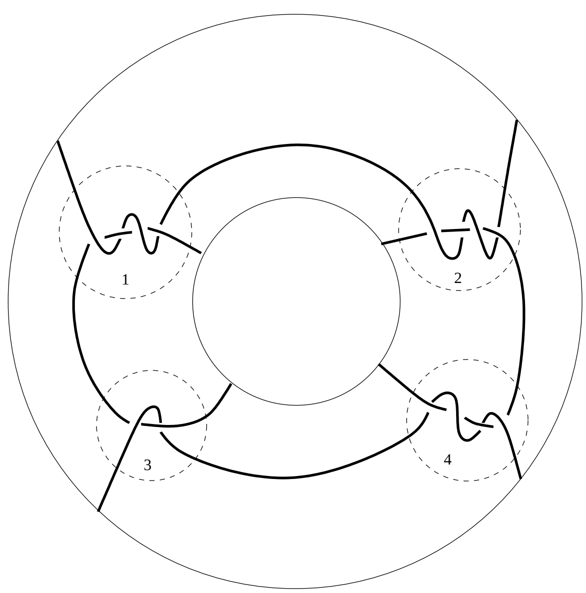


Figure 10. The spherical tangle $\boldsymbol{J}$.

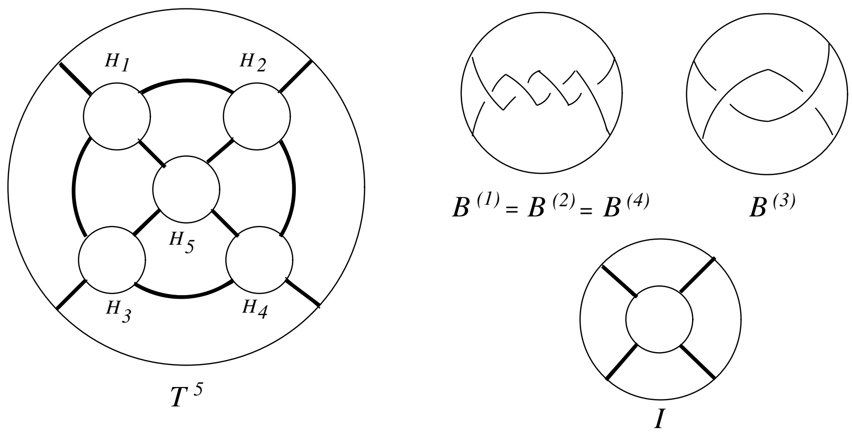


Figure 11. A decomposition of the spherical tangle $\boldsymbol{J}$.

First, let us compute $F^5(T^5)$ as the following steps:

1) The matrix

$$\begin{pmatrix} \langle T^5_{1\alpha^5_1}\rangle & \cdots & \langle T^5_{1\alpha^5_{2^5}}\rangle \\ \langle T^5_{2\alpha^5_1}\rangle & \cdots & \langle T^5_{2\alpha^5_{2^5}}\rangle \end{pmatrix}$$

is

$$\begin{pmatrix} 0010 & 1000 & 1001 & 0100 & 1001 & 0100 & 0000 & 0000 \\ 0000 & 0010 & 0000 & 1001 & 0010 & 0001 & 1001 & 0100 \end{pmatrix}.$$

2) Let

$$F^5(T^5) = \left\{ \begin{pmatrix} (-i)^{t_1} z\langle T^5_{1\alpha^5_1}\rangle & \cdots & (-i)^{t_{2^5}} z\langle T^5_{1\alpha^5_{2^5}}\rangle \\ (-i)^{t_1} iz\langle T^5_{2\alpha^5_1}\rangle & \cdots & (-i)^{t_{2^5}} iz\langle T^5_{2\alpha^5_{2^5}}\rangle \end{pmatrix} \mid z \in \Phi \right\} \cap M_{2\times 2^5}(\mathbb{Z}).$$

Then the sequence $(t_k)_{1\le k\le 2^5}$ of exponents of $-i$ is

$$0112 \quad 1223 \quad 1223 \quad 2334 \quad 1223 \quad 2334 \quad 2334 \quad 3445.$$

Therefore, by taking $z = \pm i$, we have the invariant $F^5(T^5)$ as follows.

$$F^5(T^5) = \begin{bmatrix} 0010 & 1000 & 100-1 & 0-100 & 100-1 & 0-100 & 0000 & 0000 \\ 0000 & 0010 & 0000 & 100-1 & 0010 & 000-1 & 100-1 & 0-100 \end{bmatrix}.$$

Second, we compute $[\eta^5](f(B^{(1)}), f(B^{(2)}), f(B^{(3)}), f(B^{(4)}), F(\boldsymbol{I}))$ and describe it row-by-row as follows. That is, each pair of the following means a row of the matrix $[\eta^5](f(B^{(1)}), f(B^{(2)}), f(B^{(3)}), f(B^{(4)}), F(\boldsymbol{I}))$.

$$128\ \ 0;\ 0\ \ 128;\ -32\ \ 0;\ 0\ \ -32;\ 64\ \ 0;\ 0\ \ 64;\ -16\ \ 0;\ 0\ \ -16;$$
$$-32\ \ 0;\ 0\ \ -32;\ 8\ \ 0;\ 0\ \ 8;\ -16\ \ 0;\ 0\ \ -16;\ 4\ \ 0;\ 0\ \ 4;$$
$$-32\ \ 0;\ 0\ \ -32;\ 8\ \ 0;\ 0\ \ 8;\ -16\ \ 0;\ 0\ \ -16;\ 4\ \ 0;\ 0\ \ 4;$$
$$8\ \ 0;\ 0\ \ 8;\ -2\ \ 0;\ 0\ \ -2;\ 4\ \ 0;\ 0\ \ 4;\ -1\ \ 0;\ 0\ \ -1.$$

Therefore,

$$F(\boldsymbol{J}) = F^5(T^5)[\eta^5](f(B^{(1)}), f(B^{(2)}), f(B^{(3)}), f(B^{(4)}), F(\boldsymbol{I}))$$
$$= \begin{bmatrix} -32+64-32+0+0-32+0+0 & 0+0+0-8+16+0-8+16 \\ -16-16+0+8+0+8+0+0 & 0+0-4+0-4+0+2-4 \end{bmatrix}$$
$$= \begin{bmatrix} -32 & 16 \\ -16 & -10 \end{bmatrix}.$$

Also, we have

$$\det F(\boldsymbol{J}) = (-32)(-10) - 16(-16) = 576 = 24^2.$$

Thus, $\det F(\boldsymbol{J})$ is a square of integer.

We generalize lemma 3.10 as follows.

Theorem 3.11. *Let T^5 be the 5-punctured ball tangle shown in Figure 11, and let $B^{(i)}$ be ball tangles with $f(B^{(i)}) = \begin{bmatrix} p_i \\ q_i \end{bmatrix}$ for each $i \in \{1,2,3,4\}$. Let $\boldsymbol{X} = T^5(B^{(1)}, B^{(2)}, B^{(3)}, B^{(4)}, \boldsymbol{I})$. Then*

$$F(\boldsymbol{X}) = \begin{bmatrix} p_1p_2p_3q_4+p_1p_2q_3p_4+p_1q_2p_3p_4+q_1p_2p_3p_4 & -p_1q_2p_3q_4-p_1q_2q_3p_4-q_1p_2p_3q_4-q_1p_2q_3p_4 \\ p_1p_2q_3q_4+p_1q_2q_3p_4+q_1p_2p_3q_4+q_1q_2p_3p_4 & -p_1q_2q_3q_4-q_1p_2q_3q_4-q_1q_2p_3q_4-q_1q_2q_3p_4 \end{bmatrix}.$$

Also, we have

$$\det F(\boldsymbol{X}) = (p_1q_2q_3p_4 - q_1p_2p_3q_4)^2.$$

The proof of Theorem 3.11 is quite similar as above for Lemma 3.10. Its proof is left to the reader. Notice that the determinant of $F(\boldsymbol{X})$ is also a square of integer.

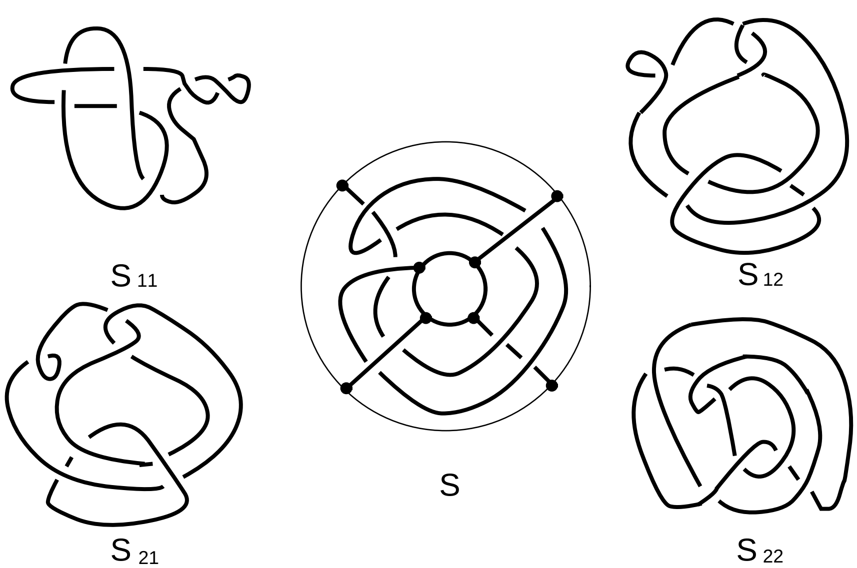


Figure 12. A spherical tangle S cannot be decomposed in terms of connect sums.

As another example, let us consider the spherical tangle diagram S in Figure 12. Remark that the Kauffman bracket is a regular isotopy invariant of link diagrams. By the definition of invariant, we have

$$F(S) = \{\begin{pmatrix} z5A^3 & -iz8A \\ iz8A & z11A^{-1} \end{pmatrix} | z \in \Phi\} \cap M_{2\times 2}(Z) = \begin{bmatrix} 5 & -8 \\ 8 & -11 \end{bmatrix}.$$

Hence, $\det F(S) = -55 - (-64) = 9$. That is, $\det F(S)$ is a square of integer. However, it seems that S can not be decomposed in terms of connect sums although we are not able to prove this fact.

For convenience, we use the following notation throughout the next section:

(1) The subscripts 1,2 of ball tangles or spherical tangles will no longer used to denote different kinds of closures. They will be used simply to distinguish different ball tangles or spherical tangles.

(2) $PM_2 = PM_{2\times 1}(\mathbb{Z})$ and $PM_{2\times 2} = PM_{2\times 2}(\mathbb{Z})$.

4. The Elementary operations on $PM_{2\times 2}$ and Coxeter groups

In this section, we introduce the group structure generated by the elementary operations on $PM_{2\times 2}$ induced by the elementary operations on $\boldsymbol{ST}$.

Let us introduce the elementary operations on $\boldsymbol{ST}$. Remark that a spherical tangle has exactly 2 holes which are inside and outside.

Definition 4.1. [4]. Let S be a spherical tangle diagram. Then

(1) S^* is the mirror image of S,

(2) S^- is the spherical tangle diagram obtained by interchanging the inside hole with the outside hole of S,

(3) S^{r_1} is the spherical tangle diagram obtained by only rotating inside hole of S $90°$ counterclockwise on the projection plane,

(4) S^{r_2} is the spherical tangle diagram obtained by only rotating outside hole of S $90°$ counterclockwise on the projection plane,

(5) S^R is the spherical tangle diagram obtained by the $90°$ rotation of S itself counterclockwise on the projection plane.

Note that $S^{r_2} = S^{-r_1-}$, $S^{r_1} = S^{-r_2-}$, and $S^R = S^{r_1 r_2} = S^{r_2 r_1}$ for each $S \in \boldsymbol{ST}$.

Lemma 4.2. [4]. *If $S \in \boldsymbol{ST}$ with the invariant $F(S) = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$, then*

(1) $F(S^*) = \begin{bmatrix} \alpha & -\gamma \\ -\beta & \delta \end{bmatrix}$, (2) $F(S^-) = \begin{bmatrix} \delta & \gamma \\ \beta & \alpha \end{bmatrix}$, (3) $F(S^{r_1}) = \begin{bmatrix} -\gamma & \alpha \\ -\delta & \beta \end{bmatrix}$,

(4) $F(S^{r_2}) = \begin{bmatrix} -\beta & -\delta \\ \alpha & \gamma \end{bmatrix}$, (5) $F(S^R) = \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix}$.

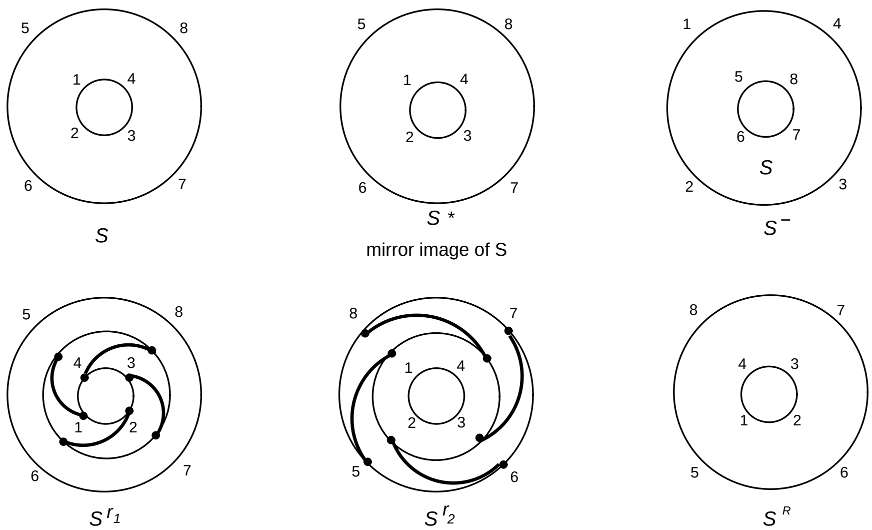


Figure 13. Elementary operations on $\boldsymbol{ST}$.

Proof. Let $S \in \boldsymbol{ST}$ with $F(S) = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$. Then there is $u \in \Phi$ such that $\langle S_{11} \rangle = \alpha u$, $\langle S_{12} \rangle = \gamma i u$, $\langle S_{21} \rangle = \beta(-i)u$, $\langle S_{22} \rangle = \delta u$. Here the link S_{ij}, $i, j \in \{1, 2\}$, is obtained by taking the numerator closure ($i = 1$) or the denominator closure ($i = 2$) of S with its hole filled by the fundamental tangle j. Therefore,

$$\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} = \begin{bmatrix} u^{-1}\alpha u & u^{-1}(-i)\gamma i u \\ u^{-1} i \beta(-i) u & u^{-1}\delta u \end{bmatrix}.$$

Now we have

(1) $\langle S^*_{11} \rangle = \alpha u^{-1}, \langle S^*_{12} \rangle = \gamma(iu)^{-1}, \langle S^*_{21} \rangle = \beta(-iu)^{-1}, \langle S^*_{22} \rangle = \delta u^{-1}$,

(2) $\langle S^-_{11} \rangle = \langle S_{22} \rangle, \langle S^-_{12} \rangle = \langle S_{12} \rangle, \langle S^-_{21} \rangle = \langle S_{21} \rangle, \langle S^-_{22} \rangle = \langle S_{11} \rangle$,

(3) $\langle S^{r_1}_{11} \rangle = \langle S_{12} \rangle, \langle S^{r_1}_{12} \rangle = \langle S_{11} \rangle, \langle S^{r_1}_{21} \rangle = \langle S_{22} \rangle, \langle S^{r_1}_{22} \rangle = \langle S_{21} \rangle$.

Hence, $F(S^*) = \begin{bmatrix} \alpha & -\gamma \\ -\beta & \delta \end{bmatrix}$, $F(S^-) = \begin{bmatrix} \delta & \gamma \\ \beta & \alpha \end{bmatrix}$, $F(S^{r_1}) = \begin{bmatrix} \gamma & -\alpha \\ \delta & -\beta \end{bmatrix} = \begin{bmatrix} -\gamma & \alpha \\ -\delta & \beta \end{bmatrix}$.

Since $S^{r_2} = S^{-r_1-}$ and $S^R = S^{r_1 r_2}$, (4) and (5) are easily proved by (2) and (3). $\square$

Like the case of ball tangle operations and invariants, it is convenient to use the following notations.

Notation: Let $\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \in PM_{2\times 2}$. Then

(1) $\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}^* = \begin{bmatrix} \alpha & -\gamma \\ -\beta & \delta \end{bmatrix}$, (2) $\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}^- = \begin{bmatrix} \delta & \gamma \\ \beta & \alpha \end{bmatrix}$, (3) $\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}^{r_1} = \begin{bmatrix} -\gamma & \alpha \\ -\delta & \beta \end{bmatrix}$,

(4) $\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}^{r_2} = \begin{bmatrix} -\beta & -\delta \\ \alpha & \gamma \end{bmatrix}$, (5) $\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}^{R} = \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix}$.

With these notations, we can write: $F(S^*) = F(S)^*, F(S^-) = F(S)^-, F(S^{r_1}) = F(S)^{r_1}, F(S^{r_2}) = F(S)^{r_2}, F(S^R) = F(S)^R$ if $S \in \boldsymbol{ST}$.

The determinant function det is well-defined on $PM_{2\times 2}$ since $\det(-A) = (-1)^2 \det A$ for each $A \in PM_{2\times 2}$.

Notice that the 5 elementary operations on $\boldsymbol{ST}$ do not change the determinant of invariants of spherical tangles.

Recall that $F(S_2 \circ S_1) = F(S_2)F(S_1)$ if $S_1, S_2 \in \boldsymbol{ST}$ (Corollary 3.6).

$$\begin{tikzcd} BT \arrow[r, "S"] \arrow[d, "f"'] & BT \arrow[d, "f"'] \\ PM_{2\times 2} \arrow[r, "F(S)"] & PM_{2\times 2} \end{tikzcd} \quad \begin{tikzcd} BT \arrow[r, "S_1"] \arrow[d, "f"'] & BT \arrow[r, "S_2"] \arrow[d, "f"'] & BT \arrow[d, "f"'] \\ PM_{2\times 2} \arrow[r, "F(S_1)"] & PM_{2\times 2} \arrow[r, "F(S_2)"] & PM_{2\times 2} \end{tikzcd} \quad \begin{tikzcd} BT \arrow[r, "S_2 \circ S_1"] \arrow[d, "f"'] & BT \arrow[d, "f"'] \\ PM_{2\times 2} \arrow[r, "F(S_2 \circ S_1)"] & PM_{2\times 2} \end{tikzcd}$$

Figure 14. Commutative diagrams of invariants.

The following lemma shows the elementary operations on the composed spherical tangle.

Lemma 4.3. [4]. *If* $S_1, S_2 \in \boldsymbol{ST}$*, then*

(1) $(S_1 \circ S_2)^* = S_1^* \circ S_2^*$, (2) $(S_1 \circ S_2)^- = S_2^- \circ S_1^-$, (3) $(S_1 \circ S_2)^{r_1} = S_1 \circ S_2^{r_1}$,
(4) $(S_1 \circ S_2)^{r_2} = S_1^{r_2} \circ S_2$, (5) $(S_1 \circ S_2)^R = S_1^R \circ S_2^R$.

Definition 4.4. An $n \times n$ matrix M is called a Coxeter matrix if $M_{ii} = 1$ and $M_{ij} = M_{ji} > 1$ for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$, where M_{ij} is the (i,j)-entry of M.

Definition 4.5. Let M be an $n \times n$ Coxeter matrix. Then a group presented by

$$\langle\, x_1, \ldots, x_n \,|\, (x_i x_j)^{M_{ij}} = 1 \text{ for all } i, j \in \{1, \ldots, n\} \,\rangle,$$

denoted by C_M, is called the Coxeter group with the Coxeter matrix M.

Let us think of the 5 elementary operations $*, -, r_1, r_2, R$ on $PM_{2\times 2}$ induced by the elementary operations on $\boldsymbol{ST}$ as functions from $PM_{2\times 2}$ to $PM_{2\times 2}$, respectively. For convenience, we use the opposite composition of functions for the binary operation. For instance, $-r_1$ means the composition $r_1 \circ -$. Recall that $S^{r_2} = S^{-r_1-}$, $S^{r_1} = S^{-r_2-}$, and $S^R = S^{r_1 r_2} = S^{r_2 r_1}$ for each $S \in \boldsymbol{ST}$ and observe the followings:

Suppose that $\begin{bmatrix}\alpha & \gamma\\ \beta & \delta\end{bmatrix} \in PM_{2\times 2}$. Then

$$\begin{bmatrix}\alpha & \gamma\\ \beta & \delta\end{bmatrix} \xrightarrow{-} \begin{bmatrix}\delta & \gamma\\ \beta & \alpha\end{bmatrix} \xrightarrow{r_1} \begin{bmatrix}-\gamma & \delta\\ -\alpha & \beta\end{bmatrix} \xrightarrow{-} \begin{bmatrix}\beta & \delta\\ -\alpha & -\gamma\end{bmatrix} \xrightarrow{r_1} \begin{bmatrix}-\delta & \beta\\ \gamma & -\alpha\end{bmatrix},$$

$$\begin{bmatrix}\alpha & \gamma\\ \beta & \delta\end{bmatrix} \xrightarrow{r_1} \begin{bmatrix}-\gamma & \alpha\\ -\delta & \beta\end{bmatrix} \xrightarrow{-} \begin{bmatrix}\beta & \alpha\\ -\delta & -\gamma\end{bmatrix} \xrightarrow{r_1} \begin{bmatrix}-\alpha & \beta\\ \gamma & -\delta\end{bmatrix} \xrightarrow{-} \begin{bmatrix}-\delta & \beta\\ \gamma & -\alpha\end{bmatrix}$$

and

$$\begin{bmatrix}\alpha & \gamma\\ \beta & \delta\end{bmatrix} \xrightarrow{-} \begin{bmatrix}\delta & \gamma\\ \beta & \alpha\end{bmatrix} \xrightarrow{*} \begin{bmatrix}\delta & -\gamma\\ -\beta & \alpha\end{bmatrix},$$

$$\begin{bmatrix}\alpha & \gamma\\ \beta & \delta\end{bmatrix} \xrightarrow{*} \begin{bmatrix}\alpha & -\gamma\\ -\beta & \delta\end{bmatrix} \xrightarrow{-} \begin{bmatrix}\delta & -\gamma\\ -\beta & \alpha\end{bmatrix}$$

and

$$\begin{bmatrix}\alpha & \gamma\\ \beta & \delta\end{bmatrix} \xrightarrow{r_1} \begin{bmatrix}-\gamma & \alpha\\ -\delta & \beta\end{bmatrix} \xrightarrow{*} \begin{bmatrix}-\gamma & -\alpha\\ \delta & \beta\end{bmatrix},$$

$$\begin{bmatrix}\alpha & \gamma\\ \beta & \delta\end{bmatrix} \xrightarrow{*} \begin{bmatrix}\alpha & -\gamma\\ -\beta & \delta\end{bmatrix} \xrightarrow{r_1} \begin{bmatrix}\gamma & \alpha\\ -\delta & -\beta\end{bmatrix}.$$

Hence, we have $-r_1 - r_1 = r_1 - r_1-$ and $-* = *-$ and $r_1* = *r_1$. Also, $--$ and r_1r_1 and $**$ are the identity function from $PM_{2\times 2}$ to $PM_{2\times 2}$. We show that the group generated by the elementary operations on $PM_{2\times 2}$ induced by those on $\boldsymbol{ST}$ has the group presentation $\langle\, x, y, z \,|\, x^2 = y^2 = z^2 = 1,\, xyxy = yxyx,\, xz = zx,\, yz = zy \,\rangle$ which is a Coxeter group.

Theorem 4.6. *The group $G(F)$ generated by the elementary operations on $PM_{2\times 2}$ induced by those on $\boldsymbol{ST}$ has the group presentation*

$$\langle\, x, y, z \,|\, x^2 = y^2 = z^2 = 1,\, xyxy = yxyx,\, xz = zx,\, yz = zy \,\rangle.$$

Furthermore, $G(F)$ is isomorphic to the Coxeter group C_M with the Coxeter matrix

$$M = \begin{pmatrix}1 & 4 & 2\\ 4 & 1 & 2\\ 2 & 2 & 1\end{pmatrix}.$$

That is,

$$G(F) = \langle\, x, y, z \,|\, x^2 = y^2 = z^2 = (xy)^4 = (yx)^4 = (xz)^2 = (zx)^2 = (yz)^2 = (zy)^2 = 1 \,\rangle.$$

Proof. Let $G = \langle\, x, y, z \,|\, x^2 = y^2 = z^2 = 1,\, xyxy = yxyx,\, xz = zx,\, yz = zy \,\rangle$. Suppose that $\phi : G \to G(F)$ is the epimorphism such that $\phi(x) = -$, $\phi(y) = r_1$, $\phi(z) = *$. We claim that $\operatorname{Ker}\phi = \{1\}$. Let $W(x, y, z)$ be a word in $\operatorname{Ker}\phi$. Then $\phi(W(x, y, z)) = W(-, r_1, *) = Id_{PM_{2\times 2}}$. Since $x^2 = y^2 = z^2 = 1$, we may assume that $W(x, y, z)$ has no consecutive letters and no inverses of letters. Since $xz = zx$ and $yz = zy$ and $z^2 = 1$, we have either $W(x, y, z) = W_1(x, y)z$ or $W(x, y, z) = W_1(x, y)$ for some word $W_1(x, y)$ in $\{x, y\}$. We may also assume that $W_1(x, y)$ has no

consecutive letters and no inverses of letters. We show that $W(x,y,z) \neq W_1(x,y)z$. If $W(x,y,z) = W_1(x,y)z$, then $W_1(-,r_1)* = Id_{PM_{2\times 2}}$. That is, $W_1(-,r_1) = *$.

Observe that

$$\begin{array}{llll} - \neq *, & -r_1 \neq *, & -r_1- \neq *, & -r_1-r_1 \neq *, \\ r_1 \neq *, & r_1- \neq *, & r_1-r_1 \neq *, & r_1-r_1- \neq *. \end{array}$$

By $-r_1-r_1 = r_1-r_1-$ and $-^2 = r_1^2 = Id_{PM_{2\times 2}}$, we have $W_1(-,r_1) \neq *$. This is a contradiction. Hence, $W(x,y,z) \neq W_1(x,y)z$. Therefore, $W(x,y,z) = W_1(x,y)$ and the number of z in $W(x,y,z)$ must be even.

Since $W_1(x,y)$ has no consecutive letters and no inverses of letters, we have either
there are $k \in \mathbb{N}\cup\{0\}$ and $R \in \{1, x, xy, xyx, xyxy, xyxyx, xyxyxy, xyxyxyx\}$
such that $W_1(x,y) = (xy)^{4k}R$ or
there are $k' \in \mathbb{N}\cup\{0\}$ and $R' \in \{1, y, yx, yxy, yxyx, yxyxy, yxyxyx, yxyxyxy\}$
such that $W_1(x,y) = (yx)^{4k'}R'$.
Also, since $W_1(x,y) = W(x,y,z) \in \operatorname{Ker}\phi$, we have either
$Id_{PM_{2\times 2}} = W_1(-,r_1) = (-r_1)^{4k}\phi(R)$ or $Id_{PM_{2\times 2}} = W_1(-,r_1) = (r_1-)^{4k'}\phi(R')$.
Similarly, as above, observe that

$$\begin{array}{llll} - \neq Id_{PM_{2\times 2}}, & -r_1 \neq Id_{PM_{2\times 2}}, & -r_1- \neq Id_{PM_{2\times 2}}, & -r_1-r_1 \neq Id_{PM_{2\times 2}}, \\ r_1 \neq Id_{PM_{2\times 2}}, & r_1- \neq Id_{PM_{2\times 2}}, & r_1-r_1 \neq Id_{PM_{2\times 2}}, & r_1-r_1- \neq Id_{PM_{2\times 2}}. \end{array}$$

Also, notice that

$$\begin{array}{ll} -r_1-r_1- = r_1-r_1, & -r_1-r_1-r_1 = r_1-, \\ -r_1-r_1-r_1- = r_1, & -r_1-r_1-r_1-r_1 = Id_{PM_{2\times 2}} \end{array}$$

and

$$\begin{array}{ll} r_1-r_1-r_1 = -r_1-, & r_1-r_1-r_1- = -r_1, \\ r_1-r_1-r_1-r_1 = -, & r_1-r_1-r_1-r_1- = Id_{PM_{2\times 2}}. \end{array}$$

Hence, we know that $\phi(R) = Id_{PM_{2\times 2}}$ if and only if $R = 1$ and $\phi(R') = Id_{PM_{2\times 2}}$ if and only if $R' = 1$.

Since $(-r_1)^{4k} = Id_{PM_{2\times 2}}$ and $(r_1-)^{4k'} = Id_{PM_{2\times 2}}$, we have $\phi(R) = Id_{PM_{2\times 2}}$ and $\phi(R') = Id_{PM_{2\times 2}}$. Hence, $R = 1$ and $R' = 1$.

Therefore, $W_1(x,y) = (xy)^{4k}$ or $W_1(x,y) = (yx)^{4k}$ for some $k \in \mathbb{N}\cup\{0\}$. Since $(xy)^4 = 1$ and $(yx)^4 = 1$, $W_1(x,y) = 1$. That is, $W(x,y,z) = 1$. We have proved $\operatorname{Ker}\phi = \{1\}$. Hence, $\phi : G \to G(F)$ is a group isomorphism and $G(F)$ has the group presentation $\langle\, x, y, z \,|\, x^2 = y^2 = z^2 = 1,\ xyxy = yxyx,\ xz = zx,\ yz = zy \,\rangle$.

Now, we show that $G(F)$ is isomorphic to C_M. Since $(xy)^2 = (yx)^2$, $(xy)^2(xy)^2 = (yx)^2(xy)^2$ and $(xy)^2(yx)^2 = (yx)^2(yx)^2$. Since $x^2 = y^2 = 1$, $(xy)^4 = (yx)^4 = 1$. Also, since $xz = zx$, $(xz)(xz) = (zx)(xz)$ and $(xz)(zx) = (zx)(zx)$. Since $x^2 = z^2 = 1$, $(xz)^2 = (zx)^2 = 1$. Similarly, since $yz = zy$, $(yz)(yz) = (zy)(yz)$ and $(yz)(zy) =$

$(zy)(zy)$. Since $y^2 = z^2 = 1$, $(yz)^2 = (zy)^2 = 1$. Hence, the consequence of relators of C_M is contained in that of $G(F)$. Conversely, Since $(xy)^4 = 1$, $(xy)^4(yx)^2 = (yx)^2$. Since $x^2 = y^2 = 1$, $(xy)^2 = (yx)^2$. Also, since $(xz)^2 = 1$, $(xz)^2(zx) = zx$. Since $x^2 = z^2 = 1$, $xz = zx$. Similarly, since $(yz)^2 = 1$, $(yz)^2(zy) = zy$. Since $y^2 = z^2 = 1$, $yz = zy$. Hence, the consequence of relators of $G(F)$ is contained in that of C_M. Thus, $G(F)$ is isomorphic to C_M. □

We have just shown that the group $G(F)$ is a Coxeter group. However, the group generated by the elementary operations on $\boldsymbol{ST}$ is not a Coxeter group because r_1 on $\boldsymbol{ST}$ has infinite order.

On the other hand, we showed the determinant of invariant of a spherical tangle is a square of integer modulo 4 in [4]. However, it seems that the determinant is a square of integer even though we don't know how to prove it yet.

APPENDIX: A guide to the nature of the calculations

We have used so complicated notations to prove Theorem 3.2 which is our first main theorem that most readers would probably feel difficult to read the proof. However, to prove it precisely, we could not help using such notations. Here, as this appendix, we try to explain such complicated notations by concrete examples with motivations to help to understand our proof of it. Also, we introduce examples for the calculation of invariant of connect sums looked like addition of fractions.

To explain the calculation process, we use elementary well-known facts, in particular, expansion of product of several polynomials by dictionary order, and finite sequences on the set $\{1, 2\}$ which are combinations of our binary digits 1 and 2.

1. Examples of finite sequences on $\{1, 2\}$:

For elements of linearly ordered set $J(n)$ by dictionary order, we write as follows.

$$\alpha^1_1 = (1), \quad \alpha^1_{2^1} = (2).$$

$$\alpha^2_1 = (11), \quad \alpha^2_2 = (12), \quad \alpha^2_3 = (21), \quad \alpha^2_{2^2} = (22).$$

$$\begin{gathered} \alpha^3_1 = (111), \quad \alpha^3_2 = (112), \quad \alpha^3_3 = (121), \quad \alpha^3_4 = (122), \\ \alpha^3_5 = (211), \quad \alpha^3_6 = (212), \quad \alpha^3_7 = (221), \quad \alpha^3_{2^3} = (222). \end{gathered}$$

$$\begin{gathered} \alpha^4_1 = (1111), \quad \alpha^4_2 = (1112), \quad \alpha^4_3 = (1121), \quad \alpha^4_4 = (1122), \\ \alpha^4_5 = (1211), \quad \alpha^4_6 = (1212), \quad \alpha^4_7 = (1221), \quad \alpha^4_8 = (1222), \\ \alpha^4_9 = (2111), \quad \alpha^4_{10} = (2112), \quad \alpha^4_{11} = (2121), \quad \alpha^4_{12} = (2122), \\ \alpha^4_{13} = (2211), \quad \alpha^4_{14} = (2212), \quad \alpha^4_{15} = (2221), \quad \alpha^4_{2^4} = (2222). \end{gathered}$$

Also, some examples of coordinates of above sequences are as follows.

$$\alpha^3_{72} = 2, \quad \alpha^2_{32} = 1, \quad \alpha^3_{2^32} = 2, \quad \alpha^4_{74} = 1, \quad \alpha^4_{2^43} = 2.$$

2. Examples to key idea (motivation to dictionary order):

When we expand a product of several polynomials, we can use the dictionary order as described. One of very complicated functions $[\eta^n]$ which is the key for the proof of Theorem 3.2 is based on the dictionary orders by which we expand the products of several polynomials.

Let us explain the following two examples which involve our idea for the main theorem.

(1) When $n = 2$, $k_1 = 2^1$, $k_2 = 2^1$,

$$\begin{aligned}
(a_1x_1 + a_{2^1}x_{2^1})(b_1y_1 + b_{2^1}y_{2^1}) &= a_1b_1x_1y_1 + a_1b_{2^1}x_1y_{2^1} + a_{2^1}b_1x_{2^1}y_1 + a_{2^1}b_{2^1}x_{2^1}y_{2^1} \\
&= \begin{pmatrix} a_1b_1 & a_1b_{2^1} & a_{2^1}b_1 & a_{2^1}b_{2^1} \end{pmatrix} \begin{pmatrix} x_1y_1 \\ x_1y_{2^1} \\ x_{2^1}y_1 \\ x_{2^1}y_{2^1} \end{pmatrix} \\
&= \xi^{2,2^1,2^1}((a_1, a_{2^1}),\ (b_1, b_{2^1}))\ \xi^{2,2^1,2^1}((x_1, x_{2^1}),\ (y_1, y_{2^1}))^{\dagger}.
\end{aligned}$$

(2) When $n = 3$, $k_1 = 2^1$, $k_2 = 2^2$, $k_3 = 2^2$,

$$\begin{aligned}
&(a_1x_1 + a_{2^1}x_{2^1})(b_1y_1 + b_2y_2 + b_3y_3 + b_{2^2}y_{2^2})(c_1z_1 + c_2z_2 + c_3z_3 + c_{2^2}z_{2^2}) \\
&= a_1b_1c_1x_1y_1z_1 + a_1b_1c_2x_1y_1z_2 + a_1b_1c_3x_1y_1z_3 + a_1b_1c_{2^2}x_1y_1z_{2^2} \\
&+ a_1b_2c_1x_1y_2z_1 + a_1b_2c_2x_1y_2z_2 + a_1b_2c_3x_1y_2z_3 + a_1b_2c_{2^2}x_1y_2z_{2^2} \\
&+ a_1b_3c_1x_1y_3z_1 + a_1b_3c_2x_1y_3z_2 + a_1b_3c_3x_1y_3z_3 + a_1b_3c_{2^2}x_1y_3z_{2^2} \\
&+ a_1b_{2^2}c_1x_1y_{2^2}z_1 + a_1b_{2^2}c_2x_1y_{2^2}z_2 + a_1b_{2^2}c_3x_1y_{2^2}z_3 + a_1b_{2^2}c_{2^2}x_1y_{2^2}z_{2^2} \\
&+ a_{2^1}b_1c_1x_{2^1}y_1z_1 + a_{2^1}b_1c_2x_{2^1}y_1z_2 + a_{2^1}b_1c_3x_{2^1}y_1z_3 + a_{2^1}b_1c_{2^2}x_{2^1}y_1z_{2^2} \\
&+ a_{2^1}b_2c_1x_{2^1}y_2z_1 + a_{2^1}b_2c_2x_{2^1}y_2z_2 + a_{2^1}b_2c_3x_{2^1}y_2z_3 + a_{2^1}b_2c_{2^2}x_{2^1}y_2z_{2^2} \\
&+ a_{2^1}b_3c_1x_{2^1}y_3z_1 + a_{2^1}b_3c_2x_{2^1}y_3z_2 + a_{2^1}b_3c_3x_{2^1}y_3z_3 + a_{2^1}b_3c_{2^2}x_{2^1}y_3z_{2^2} \\
&+ a_{2^1}b_{2^2}c_1x_{2^1}y_{2^2}z_1 + a_{2^1}b_{2^2}c_2x_{2^1}y_{2^2}z_2 + a_{2^1}b_{2^2}c_3x_{2^1}y_{2^2}z_3 + a_{2^1}b_{2^2}c_{2^2}x_{2^1}y_{2^2}z_{2^2} \\
&= \xi^{3,2^1,2^2,2^2}((a_1, a_{2^1}),\ (b_1, b_2, b_3, b_{2^2}),\ (c_1, c_2, c_3, c_{2^2}))\times \\
&\quad \xi^{3,2^1,2^2,2^2}((x_1, x_{2^1}),\ (y_1, y_2, y_3, y_{2^2}),\ (z_1, z_2, z_3, z_{2^2}))^{\dagger}.
\end{aligned}$$

3. An explanation of the proof of Theorem 3.2 by an example:

Let us consider the following example.

Suppose that $T^2, T^{2(1)}, T^{1(2)}$ are $2, 2, 1$-punctured ball tangle diagrams such that

$$F^2(T^2) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{12^2} \\ a_{21} & a_{22} & a_{23} & a_{22^2} \end{bmatrix},$$

$$F^2(T^{2(1)}) = \begin{bmatrix} b^1_{11} & b^1_{12} & b^1_{13} & b^1_{12^2} \\ b^1_{21} & b^1_{22} & b^1_{23} & b^1_{22^2} \end{bmatrix}, \; F^1(T^{1(2)}) = \begin{bmatrix} b^2_{11} & b^2_{12^1} \\ b^2_{21} & b^2_{22^1} \end{bmatrix},$$

respectively. Notice that $n = 2$, $k_1 = 2$, $k_2 = 1$.

Let $T = T^2(T^{2(1)}, T^{1(2)})$, and let $B^{(11)}, B^{(12)}, B^{(21)} \in \boldsymbol{BT}$ with

$$F^0(B^{(11)}) = \begin{bmatrix} v^{11}_1 \\ v^{11}_2 \end{bmatrix}, F^0(B^{(12)}) = \begin{bmatrix} v^{12}_1 \\ v^{12}_2 \end{bmatrix}, F^0(B^{(21)}) = \begin{bmatrix} v^{21}_1 \\ v^{21}_2 \end{bmatrix}.$$

Then $T(B^{(11)}, B^{(12)}, B^{(21)}) = T^2(T^{2(1)}(B^{(11)}, B^{(12)}), T^{1(2)}(B^{(21)}))$ and

$$\begin{aligned}
&F^0(T(B^{(11)}, B^{(12)}, B^{(21)})) = F^0(T^2(T^{2(1)}(B^{(11)}, B^{(12)}), T^{1(2)}(B^{(21)}))) \\
&= F^2(T^2)[\xi^2](F^0(T^{2(1)}(B^{(11)}, B^{(12)})), F^0(T^{1(2)}(B^{(21)}))) \\
&= F^2(T^2)[\xi^2](F^2(T^{2(1)})[\xi^2](F^0(B^{(11)}), F^0(B^{(12)})), F^1(T^{1(2)})[\xi^1](F^0(B^{(21)}))) \\
&= F^2(T^2)[\xi^2](\begin{bmatrix} b^1_{11} & b^1_{12} & b^1_{13} & b^1_{12^2} \\ b^1_{21} & b^1_{22} & b^1_{23} & b^1_{22^2} \end{bmatrix} \begin{bmatrix} \prod_{j=1}^2 v^{1j}_{\alpha^2_{1j}} \\ \prod_{j=1}^2 v^{1j}_{\alpha^2_{2j}} \\ \prod_{j=1}^2 v^{1j}_{\alpha^2_{3j}} \\ \prod_{j=1}^2 v^{1j}_{\alpha^2_{2^2 j}} \end{bmatrix}, \begin{bmatrix} b^2_{11} & b^2_{12^1} \\ b^2_{21} & b^2_{22^1} \end{bmatrix} \begin{bmatrix} \prod_{j=1}^1 v^{2j}_{\alpha^1_{1j}} \\ \prod_{j=1}^1 v^{2j}_{\alpha^1_{2^1 j}} \end{bmatrix}) \\
&= F^2(T^2)[\xi^2](\begin{bmatrix} b^1_{11} \prod_{j=1}^2 v^{1j}_{\alpha^2_{1j}} + b^1_{12} \prod_{j=1}^2 v^{1j}_{\alpha^2_{2j}} + b^1_{13} \prod_{j=1}^2 v^{1j}_{\alpha^2_{3j}} + b^1_{12^2} \prod_{j=1}^2 v^{1j}_{\alpha^2_{2^2 j}} \\ b^1_{21} \prod_{j=1}^2 v^{1j}_{\alpha^2_{1j}} + b^1_{22} \prod_{j=1}^2 v^{1j}_{\alpha^2_{2j}} + b^1_{23} \prod_{j=1}^2 v^{1j}_{\alpha^2_{3j}} + b^1_{22^2} \prod_{j=1}^2 v^{1j}_{\alpha^2_{2^2 j}} \end{bmatrix}, \\
&\begin{bmatrix} b^2_{11} \prod_{j=1}^1 v^{2j}_{\alpha^1_{1j}} + b^2_{12^1} \prod_{j=1}^1 v^{2j}_{\alpha^1_{2^1 j}} \\ b^2_{21} \prod_{j=1}^1 v^{2j}_{\alpha^1_{1j}} + b^2_{22^1} \prod_{j=1}^1 v^{2j}_{\alpha^1_{2^1 j}} \end{bmatrix}) \\
&= F^2(T^2) \begin{bmatrix} \xi^{2,2^2,2^1}((b^1_{\alpha^2_{11}1}, b^1_{\alpha^2_{11}2}, b^1_{\alpha^2_{11}3}, b^1_{\alpha^2_{11}2^2}), (b^2_{\alpha^2_{12}1}, b^2_{\alpha^2_{12}2^1})) \\ \xi^{2,2^2,2^1}((b^1_{\alpha^2_{21}1}, b^1_{\alpha^2_{21}2}, b^1_{\alpha^2_{21}3}, b^1_{\alpha^2_{21}2^2}), (b^2_{\alpha^2_{22}1}, b^2_{\alpha^2_{22}2^1})) \\ \xi^{2,2^2,2^1}((b^1_{\alpha^2_{31}1}, b^1_{\alpha^2_{31}2}, b^1_{\alpha^2_{31}3}, b^1_{\alpha^2_{31}2^2}), (b^2_{\alpha^2_{32}1}, b^2_{\alpha^2_{32}2^1})) \\ \xi^{2,2^2,2^1}((b^1_{\alpha^2_{2^2 1}1}, b^1_{\alpha^2_{2^2 1}2}, b^1_{\alpha^2_{2^2 1}3}, b^1_{\alpha^2_{2^2 1}2^2}), (b^2_{\alpha^2_{2^2 2}1}, b^2_{\alpha^2_{2^2 2}2^1})) \end{bmatrix} \times \\
&\begin{bmatrix} \xi^{2,2^2,2^1}((\prod_{j=1}^2 v^{1j}_{\alpha^2_{1j}}, \prod_{j=1}^2 v^{1j}_{\alpha^2_{2j}}, \prod_{j=1}^2 v^{1j}_{\alpha^2_{3j}}, \prod_{j=1}^2 v^{1j}_{\alpha^2_{2^2 j}}), (\prod_{j=1}^1 v^{2j}_{\alpha^1_{1j}}, \prod_{j=1}^1 v^{2j}_{\alpha^1_{2^1 j}})) \end{bmatrix}^\dagger \\
&= F^2(T^2) \begin{bmatrix} \xi^{2,2^2,2^1}((b^1_{11}, b^1_{12}, b^1_{13}, b^1_{12^2}), (b^2_{11}, b^2_{12^1})) \\ \xi^{2,2^2,2^1}((b^1_{11}, b^1_{12}, b^1_{13}, b^1_{12^2}), (b^2_{21}, b^2_{22^1})) \\ \xi^{2,2^2,2^1}((b^1_{21}, b^1_{22}, b^1_{23}, b^1_{22^2}), (b^2_{11}, b^2_{12^1})) \\ \xi^{2,2^2,2^1}((b^1_{21}, b^1_{22}, b^1_{23}, b^1_{22^2}), (b^2_{21}, b^2_{22^1})) \end{bmatrix} [\xi^{2,2^2,2^1}](\begin{bmatrix} v^{11}_1 v^{12}_1 \\ v^{11}_1 v^{12}_2 \\ v^{11}_2 v^{12}_1 \\ v^{11}_2 v^{12}_2 \end{bmatrix}, \begin{bmatrix} v^{21}_1 \\ v^{21}_2 \end{bmatrix})
\end{aligned}$$

$$= F^2(T^2)\begin{bmatrix} b^1_{11}b^2_{11} & b^1_{11}b^2_{12} & b^1_{12}b^2_{11} & b^1_{12}b^2_{12} & b^1_{13}b^2_{11} & b^1_{13}b^2_{12} & b^1_{14}b^2_{11} & b^1_{14}b^2_{12} \\ b^1_{11}b^2_{21} & b^1_{11}b^2_{22} & b^1_{12}b^2_{21} & b^1_{12}b^2_{22} & b^1_{13}b^2_{21} & b^1_{13}b^2_{22} & b^1_{14}b^2_{21} & b^1_{14}b^2_{22} \\ b^1_{21}b^2_{11} & b^1_{21}b^2_{12} & b^1_{22}b^2_{11} & b^1_{22}b^2_{12} & b^1_{23}b^2_{11} & b^1_{23}b^2_{12} & b^1_{24}b^2_{11} & b^1_{24}b^2_{12} \\ b^1_{21}b^2_{21} & b^1_{21}b^2_{22} & b^1_{22}b^2_{21} & b^1_{22}b^2_{22} & b^1_{23}b^2_{21} & b^1_{23}b^2_{22} & b^1_{24}b^2_{21} & b^1_{24}b^2_{22} \end{bmatrix}\begin{bmatrix} v^{11}_1 v^{12}_1 v^{21}_1 \\ v^{11}_1 v^{12}_1 v^{21}_2 \\ v^{11}_1 v^{12}_2 v^{21}_1 \\ v^{11}_1 v^{12}_2 v^{21}_2 \\ v^{11}_2 v^{12}_1 v^{21}_1 \\ v^{11}_2 v^{12}_1 v^{21}_2 \\ v^{11}_2 v^{12}_2 v^{21}_1 \\ v^{11}_2 v^{12}_2 v^{21}_2 \end{bmatrix}$$

$$= F^2(T^2)[\eta^2](F^2(T^{2(1)}), F^1(T^{1(2)}))[\xi^{2+1}](F^0(B^{(11)}), F^0(B^{(12)}), F^0(B^{(21)}))$$
$$= F^{2+1}(T)[\xi^{2+1}](F^0(B^{(11)}), F^0(B^{(12)}), F^0(B^{(21)})).$$

By Lemma 3.1, we conclude that

$$F^3(T) = F^3(T^2(T^{2(1)}, T^{1(2)})) = F^2(T^2)[\eta^2](F^2(T^{2(1)}), F^1(T^{1(2)})) =$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}\begin{bmatrix} b^1_{11}b^2_{11} & b^1_{11}b^2_{12} & b^1_{12}b^2_{11} & b^1_{12}b^2_{12} & b^1_{13}b^2_{11} & b^1_{13}b^2_{12} & b^1_{14}b^2_{11} & b^1_{14}b^2_{12} \\ b^1_{11}b^2_{21} & b^1_{11}b^2_{22} & b^1_{12}b^2_{21} & b^1_{12}b^2_{22} & b^1_{13}b^2_{21} & b^1_{13}b^2_{22} & b^1_{14}b^2_{21} & b^1_{14}b^2_{22} \\ b^1_{21}b^2_{11} & b^1_{21}b^2_{12} & b^1_{22}b^2_{11} & b^1_{22}b^2_{12} & b^1_{23}b^2_{11} & b^1_{23}b^2_{12} & b^1_{24}b^2_{11} & b^1_{24}b^2_{12} \\ b^1_{21}b^2_{21} & b^1_{21}b^2_{22} & b^1_{22}b^2_{21} & b^1_{22}b^2_{22} & b^1_{23}b^2_{21} & b^1_{23}b^2_{22} & b^1_{24}b^2_{21} & b^1_{24}b^2_{22} \end{bmatrix}.$$

Now, let us explain why Lemma 3.1 is required to complete this example.

Suppose that $A = F^3(T^2(T^{2(1)}, T^{1(2)}))$ and $B = F^2(T^2)[\eta^2](F^2(T^{2(1)}), F^1(T^{1(2)}))$. Then A and B are matrices in $PM_{2\times 2^3}(\mathbb{Z})$. In order to show $A = B$, we have shown that $AX = BX$ for each

$$X \in \{[\xi^3](F^0(B^{(11)}), F^0(B^{(12)}), F^0(B^{(21)})) | B^{(11)}, B^{(12)}, B^{(21)} \in \boldsymbol{BT}\}.$$

In [4], we proved that the 0-punctured ball tangle invariant $F^0 : \boldsymbol{BT} \to PM_{2\times 1}(\mathbb{Z})$ is surjective. Note that $PM_{2\times 1}(\mathbb{Z}) = P\mathbb{Z}^{2\dagger}$. So we have

$$\begin{aligned}\{[\xi^3](F^0(B^{(11)}), F^0(B^{(12)}), F^0(B^{(21)})) | B^{(11)}, B^{(12)}, B^{(21)} \in \boldsymbol{BT}\}\\ = [\xi^3](P\mathbb{Z}^{2\dagger} \times P\mathbb{Z}^{2\dagger} \times P\mathbb{Z}^{2\dagger}).\end{aligned}$$

Fortunately, we have ball tangles $B^{(1)}$, $B^{(2)}$, $B^{(3)}$ whose invariants are $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$, respectively (See Figure 8). Note that $[\xi^3](P\mathbb{Z}^{2\dagger}\times P\mathbb{Z}^{2\dagger}\times P\mathbb{Z}^{2\dagger}) \subsetneqq P\mathbb{Z}^{8\dagger}$. For example, we easily know that $\begin{bmatrix}0 & 1 & 1 & 1 & 1 & 1 & 1 & 1\end{bmatrix}^\dagger \notin [\xi^3](P\mathbb{Z}^{2\dagger} \times P\mathbb{Z}^{2\dagger} \times P\mathbb{Z}^{2\dagger})$.

Lemma 3.1 says that we have only to show that $AX = BX$ for each

$$X \in [\xi^3](\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\} \times \{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\} \times \{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\}).$$

That is, we have only to check the following 27 column vectors in $P\mathbb{Z}^{8\dagger}$:

$[1\,0\,0\,0\,0\,0\,0\,0]^\dagger, [0\,1\,0\,0\,0\,0\,0\,0]^\dagger, [1\,1\,0\,0\,0\,0\,0\,0]^\dagger, [0\,0\,1\,0\,0\,0\,0\,0]^\dagger, [0\,0\,0\,1\,0\,0\,0\,0]^\dagger,$

$[0\,0\,1\,1\,0\,0\,0\,0]^\dagger, [1\,0\,1\,0\,0\,0\,0\,0]^\dagger, [0\,1\,0\,1\,0\,0\,0\,0]^\dagger, [1\,1\,1\,1\,0\,0\,0\,0]^\dagger, [0\,0\,0\,0\,1\,0\,0\,0]^\dagger,$

$[0\,0\,0\,0\,0\,1\,0\,0]^\dagger, [0\,0\,0\,0\,1\,1\,0\,0]^\dagger, [0\,0\,0\,0\,0\,0\,1\,0]^\dagger, [0\,0\,0\,0\,0\,0\,0\,1]^\dagger, [0\,0\,0\,0\,0\,0\,1\,1]^\dagger,$

$$[\,0\,0\,0\,0\,1\,0\,1\,0\,]^\dagger, [\,0\,0\,0\,0\,0\,1\,0\,1\,]^\dagger, [\,0\,0\,0\,0\,1\,1\,1\,1\,]^\dagger, [\,1\,0\,0\,0\,1\,0\,0\,0\,]^\dagger, [\,0\,1\,0\,0\,0\,1\,0\,0\,]^\dagger,$$
$$[\,1\,1\,0\,0\,1\,1\,0\,0\,]^\dagger, [\,0\,0\,1\,0\,0\,0\,1\,0\,]^\dagger, [\,0\,0\,0\,1\,0\,0\,0\,1\,]^\dagger, [\,0\,0\,1\,1\,0\,0\,1\,1\,]^\dagger, [\,1\,0\,1\,0\,1\,0\,1\,0\,]^\dagger,$$
$$[\,0\,1\,0\,1\,0\,1\,0\,1\,]^\dagger, [\,1\,1\,1\,1\,1\,1\,1\,1\,]^\dagger.$$

4. The invariant of connect sums looked like addition of fractions:

Recall Corollary 3.7 to explain the calculation process by Theorem 3.3 which is our second main Theorem. If $B^{(1)}, B^{(2)} \in \boldsymbol{BT}$ with $F^0(B^{(1)}) = \begin{bmatrix} p \\ q \end{bmatrix}$ and $F^0(B^{(2)}) = \begin{bmatrix} r \\ s \end{bmatrix}$, then (1) $F^0(B^{(1)} +_h B^{(2)}) = \begin{bmatrix} ps+qr \\ qs \end{bmatrix}$ (Krebes [5]), (2) $F^0(B^{(1)} +_v B^{(2)}) = \begin{bmatrix} pr \\ qr+ps \end{bmatrix}$.

Consider the addition of fractions:

$$\frac{p}{q} + \frac{r}{s} = \frac{ps+qr}{qs}, \quad \frac{1}{\frac{1}{\frac{p}{q}} + \frac{1}{\frac{r}{s}}} = \frac{pr}{ps+qr}.$$

They look like the invariant of connect sums of ball tangles.

Let us consider the following example.

Suppose that $T^{2(1)}, T^{1(2)}$ are $2,1$-punctured ball tangle diagrams such that

$$F^2(T^{2(1)}) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{12^2} \\ a_{21} & a_{22} & a_{23} & a_{22^2} \end{bmatrix}, \; F^1(T^{1(2)}) = \begin{bmatrix} b_{11} & b_{12^1} \\ b_{21} & b_{22^1} \end{bmatrix},$$

respectively. Notice that $k_1 = 2$, $k_2 = 1$.

Let $T = T^{2(1)} +_h T^{1(2)}$, and let $B^{(11)}, B^{(12)}, B^{(21)} \in \boldsymbol{BT}$ with

$$F^0(B^{(11)}) = \begin{bmatrix} v_1^{11} \\ v_2^{11} \end{bmatrix}, F^0(B^{(12)}) = \begin{bmatrix} v_1^{12} \\ v_2^{12} \end{bmatrix}, F^0(B^{(21)}) = \begin{bmatrix} v_1^{21} \\ v_2^{21} \end{bmatrix}.$$

Then $T(B^{(11)}, B^{(12)}, B^{(21)}) = T^{2(1)}(B^{(11)}, B^{(12)}) +_h T^{1(2)}(B^{(21)})$ and

$$\begin{aligned}
&F^0(T(B^{(11)}, B^{(12)}, B^{(21)})) = F^0(T^{2(1)}(B^{(11)}, B^{(12)}) +_h T^{1(2)}(B^{(21)})) \\
&= F^0(T^{2(1)}(B^{(11)}, B^{(12)})) +_h F^0(T^{1(2)}(B^{(21)})) \\
&= F^2(T^{2(1)})[\xi^2](F^0(B^{(11)}), F^0(B^{(12)})) +_h F^1(T^{1(2)})[\xi^1](F^0(B^{(21)})) \\
&= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{12^2} \\ a_{21} & a_{22} & a_{23} & a_{22^2} \end{bmatrix} \begin{bmatrix} \prod_{j=1}^2 v^{1j}_{\alpha^2_{1j}} \\ \prod_{j=1}^2 v^{1j}_{\alpha^2_{2j}} \\ \prod_{j=1}^2 v^{1j}_{\alpha^2_{3j}} \\ \prod_{j=1}^2 v^{1j}_{\alpha^2_{2^2 j}} \end{bmatrix} +_h \begin{bmatrix} b_{11} & b_{12^1} \\ b_{21} & b_{22^1} \end{bmatrix} \begin{bmatrix} \prod_{j=1}^1 v^{2j}_{\alpha^1_{1j}} \\ \prod_{j=1}^1 v^{2j}_{\alpha^1_{2^1 j}} \end{bmatrix} \\
&= \begin{bmatrix} a_{11} \prod_{j=1}^2 v^{1j}_{\alpha^2_{1j}} + a_{12} \prod_{j=1}^2 v^{1j}_{\alpha^2_{2j}} + a_{13} \prod_{j=1}^2 v^{1j}_{\alpha^2_{3j}} + a_{12^2} \prod_{j=1}^2 v^{1j}_{\alpha^2_{2^2 j}} \\ a_{21} \prod_{j=1}^2 v^{1j}_{\alpha^2_{1j}} + a_{22} \prod_{j=1}^2 v^{1j}_{\alpha^2_{2j}} + a_{23} \prod_{j=1}^2 v^{1j}_{\alpha^2_{3j}} + a_{22^2} \prod_{j=1}^2 v^{1j}_{\alpha^2_{2^2 j}} \end{bmatrix} +_h
\end{aligned}$$

$$\begin{bmatrix} b_{11}\prod_{j=1}^{1} v^{2j}_{\alpha^1_{1j}} + b_{12^1}\prod_{j=1}^{1} v^{2j}_{\alpha^1_{2^1j}} \\ b_{21}\prod_{j=1}^{1} v^{2j}_{\alpha^1_{1j}} + b_{22^1}\prod_{j=1}^{1} v^{2j}_{\alpha^1_{2^1j}} \end{bmatrix}$$

$$= \begin{bmatrix} \xi^{2,2^2,2^1}((a_{11},a_{12},a_{13},a_{12^2}),(b_{21},b_{22^1})) + \xi^{2,2^2,2^1}((a_{21},a_{22},a_{23},a_{22^2}),(b_{11},b_{12^1})) \\ \xi^{2,2^2,2^1}((a_{21},a_{22},a_{23},a_{22^2}),(b_{21},b_{22^1})) \end{bmatrix} \times$$

$$\left[\xi^{2,2^2,2^1}((\prod_{j=1}^{2} v^{1j}_{\alpha^2_{1j}}, \prod_{j=1}^{2} v^{1j}_{\alpha^2_{2j}}, \prod_{j=1}^{2} v^{1j}_{\alpha^2_{3j}}, \prod_{j=1}^{2} v^{1j}_{\alpha^2_{2^2j}}),(\prod_{j=1}^{1} v^{2j}_{\alpha^1_{1j}}, \prod_{j=1}^{1} v^{2j}_{\alpha^1_{2^1j}}))\right]^{\dagger}$$

$$= [\tfrac{a_{11}b_{21}+a_{21}b_{11}}{a_{21}b_{21}}\ \tfrac{a_{11}b_{22^1}+a_{21}b_{12^1}}{a_{21}b_{22^1}}\ \tfrac{a_{12}b_{21}+a_{22}b_{11}}{a_{22}b_{21}}\ \tfrac{a_{12}b_{22^1}+a_{22}b_{12^1}}{a_{22}b_{22^1}}$$

$$\tfrac{a_{13}b_{21}+a_{23}b_{11}}{a_{23}b_{21}}\ \tfrac{a_{13}b_{22^1}+a_{23}b_{12^1}}{a_{23}b_{22^1}}\ \tfrac{a_{12^2}b_{21}+a_{22^2}b_{11}}{a_{22^2}b_{21}}\ \tfrac{a_{12^2}b_{22^1}+a_{22^2}b_{12^1}}{a_{22^2}b_{22^1}}][\xi^{2,2^2,2^1}](\begin{bmatrix} v^{11}_1 v^{12}_1 \\ v^{11}_1 v^{12}_2 \\ v^{11}_2 v^{12}_1 \\ v^{11}_2 v^{12}_2 \end{bmatrix}, \begin{bmatrix} v^{21}_1 \\ v^{21}_2 \end{bmatrix})$$

$$= [\tfrac{a_{11}b_{21}+a_{21}b_{11}}{a_{21}b_{21}}\ \tfrac{a_{11}b_{22^1}+a_{21}b_{12^1}}{a_{21}b_{22^1}}\ \tfrac{a_{12}b_{21}+a_{22}b_{11}}{a_{22}b_{21}}\ \tfrac{a_{12}b_{22^1}+a_{22}b_{12^1}}{a_{22}b_{22^1}}$$

$$\tfrac{a_{13}b_{21}+a_{23}b_{11}}{a_{23}b_{21}}\ \tfrac{a_{13}b_{22^1}+a_{23}b_{12^1}}{a_{23}b_{22^1}}\ \tfrac{a_{12^2}b_{21}+a_{22^2}b_{11}}{a_{22^2}b_{21}}\ \tfrac{a_{12^2}b_{22^1}+a_{22^2}b_{12^1}}{a_{22^2}b_{22^1}}] \begin{bmatrix} v^{11}_1 v^{12}_1 v^{21}_1 \\ v^{11}_1 v^{12}_1 v^{21}_2 \\ v^{11}_1 v^{12}_2 v^{21}_1 \\ v^{11}_1 v^{12}_2 v^{21}_2 \\ v^{11}_2 v^{12}_1 v^{21}_1 \\ v^{11}_2 v^{12}_1 v^{21}_2 \\ v^{11}_2 v^{12}_2 v^{21}_1 \\ v^{11}_2 v^{12}_2 v^{21}_2 \end{bmatrix}$$

$$= F^{2+1}(T)[\xi^{2+1}](F^0(B^{(11)}), F^0(B^{(12)}), F^0(B^{(21)})).$$

Therefore, by Lemma 3.1, we have

$$F^{2+1}(T^{2(1)} +_h T^{1(2)}) = [\tfrac{a_{11}b_{21}+a_{21}b_{11}}{a_{21}b_{21}}\ \tfrac{a_{11}b_{22^1}+a_{21}b_{12^1}}{a_{21}b_{22^1}}\ \tfrac{a_{12}b_{21}+a_{22}b_{11}}{a_{22}b_{21}}\ \tfrac{a_{12}b_{22^1}+a_{22}b_{12^1}}{a_{22}b_{22^1}}$$

$$\tfrac{a_{13}b_{21}+a_{23}b_{11}}{a_{23}b_{21}}\ \tfrac{a_{13}b_{22^1}+a_{23}b_{12^1}}{a_{23}b_{22^1}}\ \tfrac{a_{12^2}b_{21}+a_{22^2}b_{11}}{a_{22^2}b_{21}}\ \tfrac{a_{12^2}b_{22^1}+a_{22^2}b_{12^1}}{a_{22^2}b_{22^1}}].$$

Also, we can write

$$F^{2+1}(T^{2(1)} +_h T^{1(2)}) = \left[\left(\begin{pmatrix} a_{1i}b_{2j} + a_{2i}b_{1j} \\ a_{2i}b_{2j} \end{pmatrix}_{j=1,2^1}\right)_{i=1,2,3,2^2}\right].$$

Similarly, we can show the following formula for the vertical connect sum.

$$F^{2+1}(T^{2(1)} +_v T^{1(2)}) = \left[\left(\begin{pmatrix} a_{1i}b_{1j} \\ a_{2i}b_{1j} + a_{1i}b_{2j} \end{pmatrix}_{j=1,2^1}\right)_{i=1,2,3,2^2}\right].$$

Notice that the addition of fractions still plays an important role in the calculation process of the invariant of connect sums of punctured ball tangles.

We have tried to make our main theorems easier by concrete examples. Even though we have used very complicated notations, we think of our method as a kind of primitive applications of dictionary orders.

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